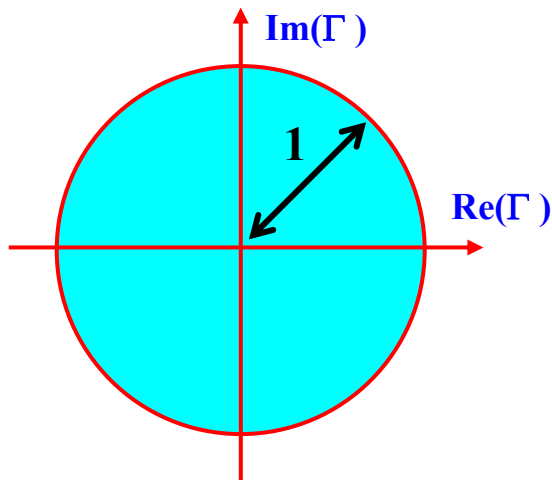


Smith Chart

The Smith chart is one of the most useful **graphical tools** for high frequency circuit applications. The chart provides a clever way to visualize complex functions and it continues to endure popularity, decades after its original conception.

From a mathematical point of view, the Smith chart is a 4-D representation of all possible complex impedances with respect to coordinates defined by the complex reflection coefficient.



The domain of definition of the reflection coefficient for a loss-less line is a circle of unitary radius in the complex plane. This is also the domain of the Smith chart.

In the case of a general lossy line, the reflection coefficient might have magnitude larger than one, due to the complex characteristic impedance, requiring an extended Smith chart.

The goal of the Smith chart is to identify **all possible impedances** on the domain of existence of the reflection coefficient. To do so, we start from the general definition of **line impedance** (which is equally applicable to a load impedance when $d=0$)

$$Z(d) = \frac{V(d)}{I(d)} = Z_0 \frac{1 + \Gamma(d)}{1 - \Gamma(d)}$$

This provides the complex function $Z(d) = f\{\text{Re}(\Gamma), \text{Im}(\Gamma)\}$ that we want to graph. It is obvious that the result would be applicable only to lines with exactly characteristic impedance Z_0 .

In order to obtain **universal** curves, we introduce the concept of **normalized impedance**

$$z_n(d) = \frac{Z(d)}{Z_0} = \frac{1 + \Gamma(d)}{1 - \Gamma(d)}$$

The **normalized impedance** is represented on the **Smith chart** by using families of curves that identify the **normalized resistance r** (real part) and the **normalized reactance x** (imaginary part)

$$z_n(d) = \text{Re}(z_n) + j \text{Im}(z_n) = r + jx$$

Let's represent the **reflection coefficient** in terms of its coordinates

$$\Gamma(d) = \text{Re}(\Gamma) + j \text{Im}(\Gamma)$$

Now we can write

$$\begin{aligned} r + jx &= \frac{1 + \text{Re}(\Gamma) + j \text{Im}(\Gamma)}{1 - \text{Re}(\Gamma) - j \text{Im}(\Gamma)} \\ &= \frac{1 - \text{Re}^2(\Gamma) - \text{Im}^2(\Gamma) + j2 \text{Im}(\Gamma)}{(1 - \text{Re}(\Gamma))^2 + \text{Im}^2(\Gamma)} \end{aligned}$$

The **real part** gives

$$r = \frac{1 - \operatorname{Re}^2(\Gamma) - \operatorname{Im}^2(\Gamma)}{(1 - \operatorname{Re}(\Gamma))^2 + \operatorname{Im}^2(\Gamma)}$$

Add a quantity equal to zero

$$r(\operatorname{Re}(\Gamma) - 1)^2 + (\operatorname{Re}^2(\Gamma) - 1) + r\operatorname{Im}^2(\Gamma) + \operatorname{Im}^2(\Gamma) + \underbrace{\frac{1}{1+r} - \frac{1}{1+r}}_{=0} = 0$$

$$\left[r(\operatorname{Re}(\Gamma) - 1)^2 + (\operatorname{Re}^2(\Gamma) - 1) + \frac{1}{1+r} \right] + (1+r)\operatorname{Im}^2(\Gamma) = \frac{1}{1+r}$$

$$(1+r) \left[\operatorname{Re}^2(\Gamma) - 2\operatorname{Re}(\Gamma)\frac{r}{1+r} + \frac{r^2}{(1+r)^2} \right] + (1+r)\operatorname{Im}^2(\Gamma) = \frac{1}{1+r}$$

$$\Rightarrow \left[\operatorname{Re}(\Gamma) - \frac{r}{1+r} \right]^2 + \operatorname{Im}^2(\Gamma) = \left(\frac{1}{1+r} \right)^2$$

Equation of a circle

The **imaginary part** gives

$$x = \frac{2 \operatorname{Im}(\Gamma)}{(1 - \operatorname{Re}(\Gamma))^2 + \operatorname{Im}^2(\Gamma)}$$

Multiply by x and add a quantity equal to zero

$$x^2 \left[(1 - \operatorname{Re}(\Gamma))^2 + \operatorname{Im}^2(\Gamma) \right] - 2x \operatorname{Im}(\Gamma) + \underbrace{1 - 1}_{= 0} = 0$$

$$\left[(1 - \operatorname{Re}(\Gamma))^2 + \operatorname{Im}^2(\Gamma) \right] - \frac{2}{x} \operatorname{Im}(\Gamma) + \frac{1}{x^2} = \frac{1}{x^2}$$

$$(1 - \operatorname{Re}(\Gamma))^2 + \left[\operatorname{Im}^2(\Gamma) - \frac{2}{x} \operatorname{Im}(\Gamma) + \frac{1}{x^2} \right] = \frac{1}{x^2}$$

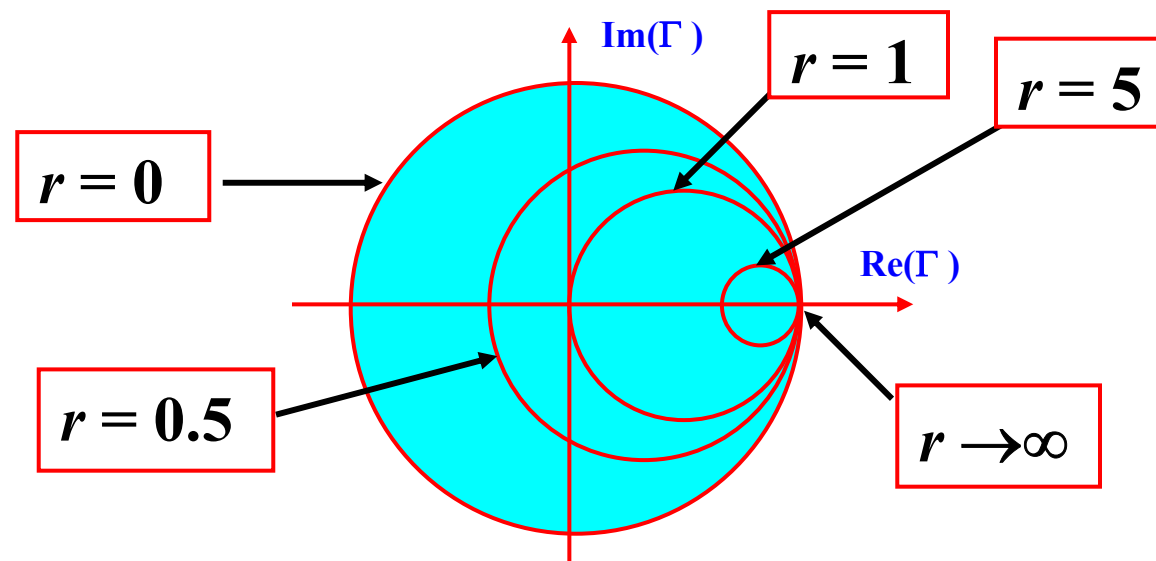
$$\Rightarrow (\operatorname{Re}(\Gamma) - 1)^2 + \left[\operatorname{Im}(\Gamma) - \frac{1}{x} \right]^2 = \frac{1}{x^2}$$

Equation of a circle

The result for the **real part** indicates that on the complex plane with coordinates $(\text{Re}(\Gamma), \text{Im}(\Gamma))$ all the possible impedances with a given normalized resistance r are found on a **circle** with

$$\text{Center} = \left\{ \frac{r}{1+r}, 0 \right\} \qquad \text{Radius} = \frac{1}{1+r}$$

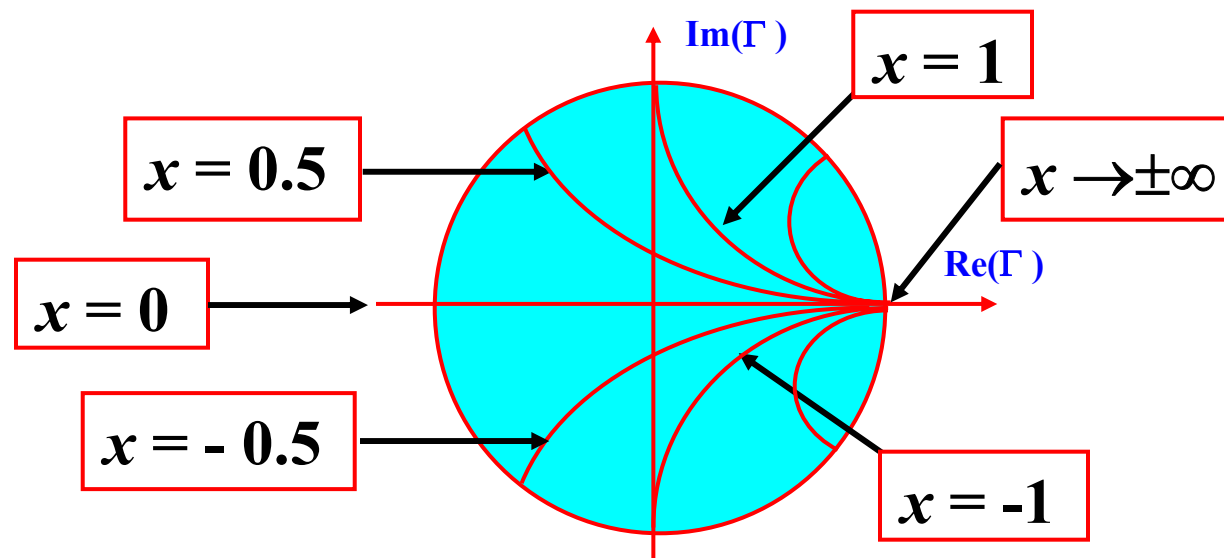
As the normalized resistance r varies from 0 to ∞ , we obtain a family of circles completely contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



The result for the **imaginary part** indicates that on the complex plane with coordinates $(\text{Re}(\Gamma), \text{Im}(\Gamma))$ all the possible impedances with a given normalized reactance x are found on a **circle** with

$$\text{Center} = \left\{ 1, \frac{1}{x} \right\} \qquad \text{Radius} = \frac{1}{x}$$

As the normalized reactance x varies from $-\infty$ to ∞ , we obtain a family of arcs contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



Basic Smith Chart techniques for loss-less transmission lines

- Given $Z(d) \Rightarrow$ Find $\Gamma(d)$
 Given $\Gamma(d) \Rightarrow$ Find $Z(d)$
- Given Γ_R and $Z_R \Rightarrow$ Find $\Gamma(d)$ and $Z(d)$
 Given $\Gamma(d)$ and $Z(d) \Rightarrow$ Find Γ_R and Z_R
- Find d_{\max} and d_{\min} (maximum and minimum locations for the voltage standing wave pattern)
- Find the Voltage Standing Wave Ratio (VSWR)
- Given $Z(d) \Rightarrow$ Find $Y(d)$
 Given $Y(d) \Rightarrow$ Find $Z(d)$

Given $Z(d)$ \Rightarrow Find $\Gamma(d)$

1. Normalize the impedance

$$z_n(d) = \frac{Z(d)}{Z_0} = \frac{R}{Z_0} + j \frac{X}{Z_0} = r + j x$$

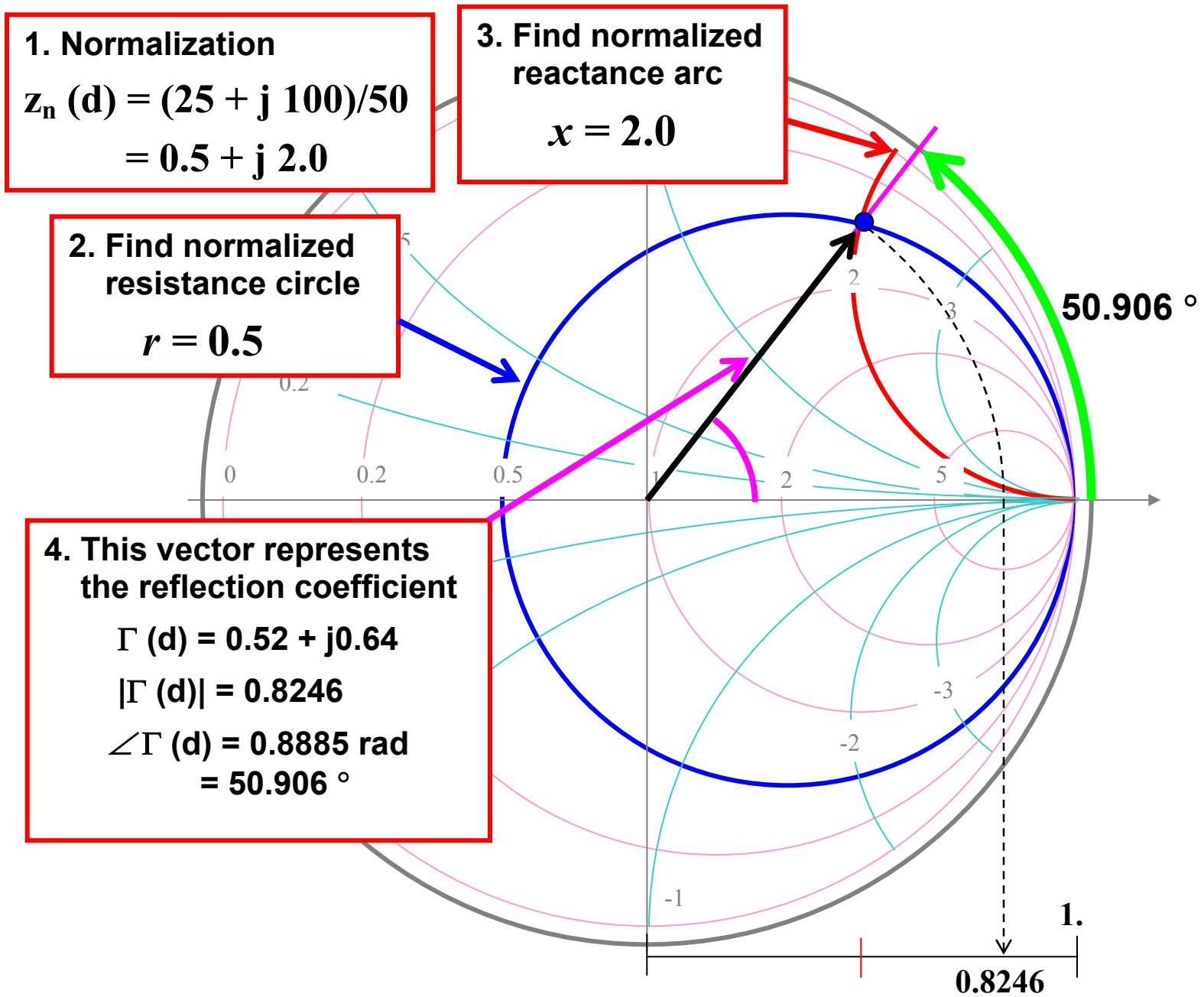
2. Find the circle of constant normalized resistance r

3. Find the arc of constant normalized reactance x

4. The intersection of the two curves indicates the reflection coefficient in the complex plane. The chart provides directly the magnitude and the phase angle of $\Gamma(d)$

Example: Find $\Gamma(d)$, given

$$Z(d) = 25 + j 100 \Omega \quad \text{with} \quad Z_0 = 50 \Omega$$



Given $\Gamma(d)$ \Rightarrow Find $Z(d)$

- 1. Determine the complex point representing the given reflection coefficient $\Gamma(d)$ on the chart.**
- 2. Read the values of the normalized resistance r and of the normalized reactance x that correspond to the reflection coefficient point.**
- 3. The normalized impedance is**

$$z_n(d) = r + jx$$

and the actual impedance is

$$Z(d) = Z_0 z_n(d) = Z_0 (r + jx) = Z_0 r + j Z_0 x$$

Given Γ_R and $Z_R \iff$ Find $\Gamma(d)$ and $Z(d)$

NOTE: the **magnitude** of the **reflection coefficient** is **constant** along a loss-less transmission line terminated by a specified load, since

$$|\Gamma(d)| = |\Gamma_R \exp(-j2\beta d)| = |\Gamma_R|$$

Therefore, on the complex plane, a **circle** with center at the **origin** and radius $|\Gamma_R|$ represents all possible reflection coefficients found along the transmission line. When the **circle of constant magnitude** of the **reflection coefficient** is drawn on the Smith chart, one can determine the values of the line **impedance** at **any location**.

The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.

2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.
3. Starting from the point representing the load, travel on the circle in the **clockwise** direction, by an angle

$$\theta = 2 \beta d = 2 \frac{2\pi}{\lambda} d$$

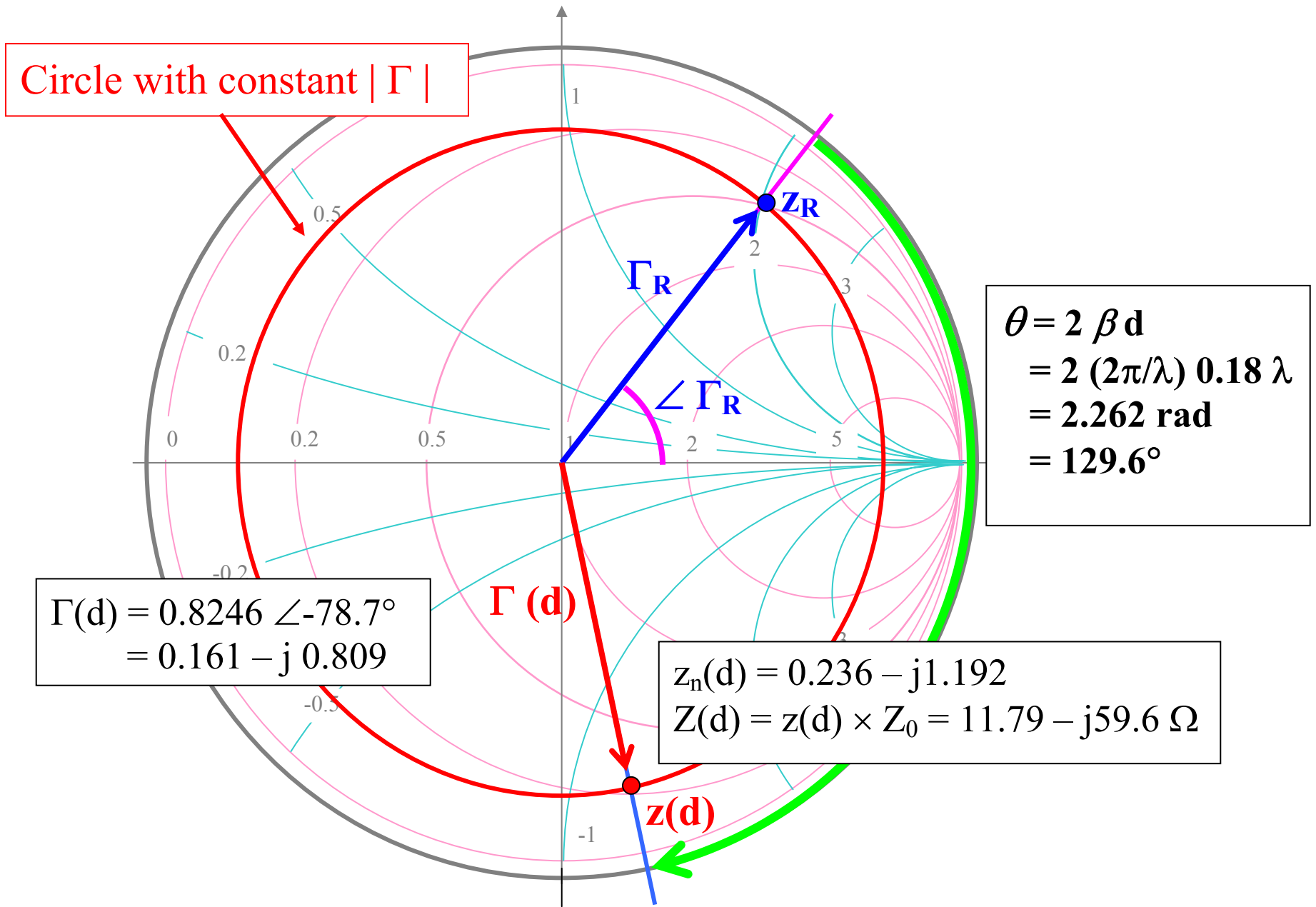
4. The new location on the chart corresponds to location **d** on the transmission line. Here, the values of $\Gamma(d)$ and $Z(d)$ can be read from the chart as before.

Example: Given

$$Z_R = 25 + j 100 \Omega \quad \text{with} \quad Z_0 = 50 \Omega$$

find

$$Z(d) \quad \text{and} \quad \Gamma(d) \quad \text{for} \quad d = 0.18\lambda$$



Given Γ_R and $Z_R \Rightarrow$ Find d_{\max} and d_{\min}

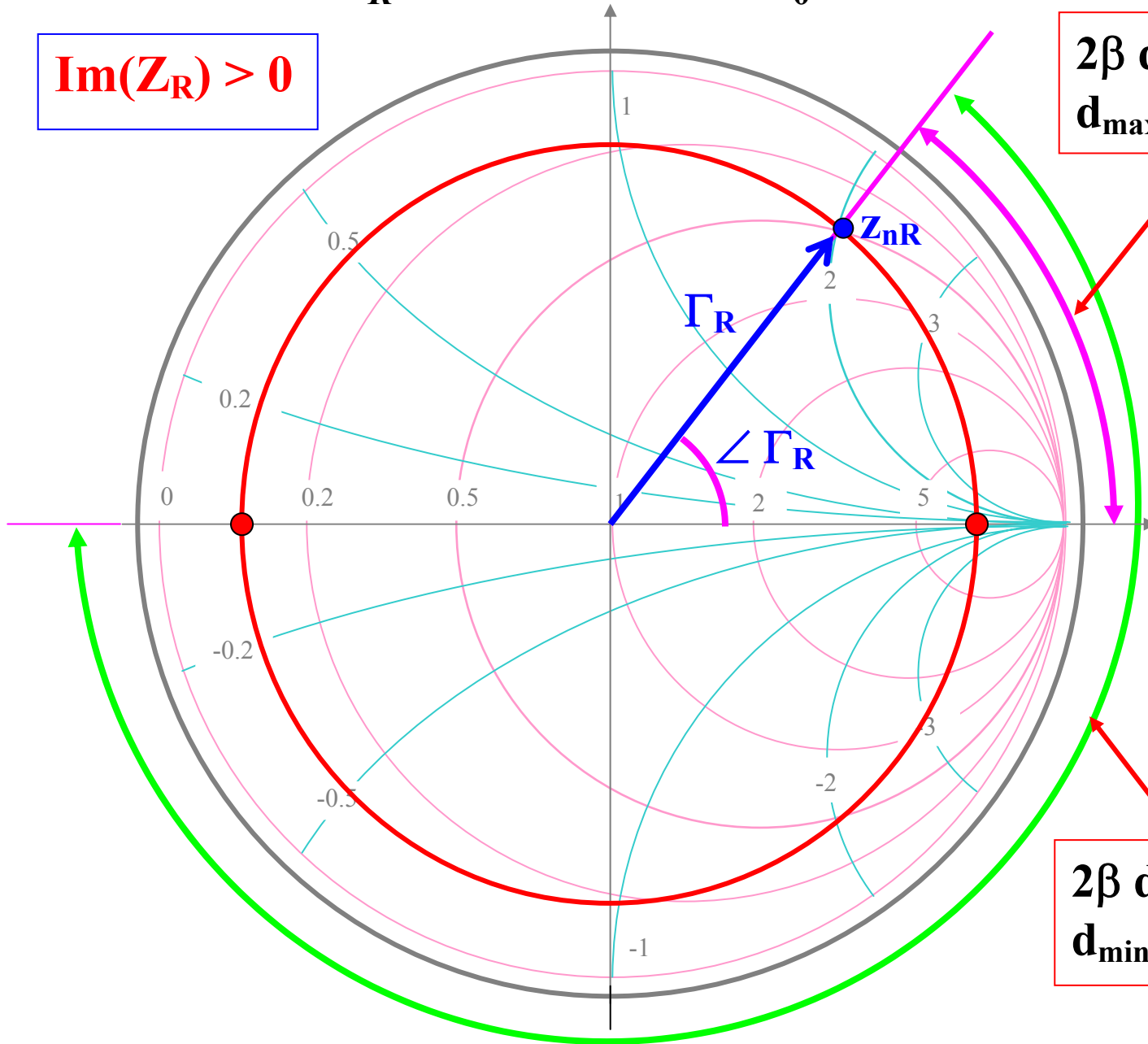
1. Identify on the Smith chart the load reflection coefficient Γ_R or the normalized load impedance Z_R .
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$. The circle intersects the real axis of the reflection coefficient at two points which identify d_{\max} (when $\Gamma(d) = \text{Real positive}$) and d_{\min} (when $\Gamma(d) = \text{Real negative}$)
3. A commercial Smith chart provides an outer graduation where the distances normalized to the wavelength can be read directly. The angles, between the vector Γ_R and the real axis, also provide a way to compute d_{\max} and d_{\min} .

Example: Find d_{\max} and d_{\min} for

$$Z_R = 25 + j100 \Omega \ ; \ Z_R = 25 - j100\Omega \quad (Z_0 = 50 \Omega)$$

$$Z_R = 25 + j 100 \Omega \quad (Z_0 = 50 \Omega)$$

$\text{Im}(Z_R) > 0$



$2\beta d_{\max} = 50.9^\circ$
 $d_{\max} = 0.0707\lambda$

$2\beta d_{\min} = 230.9^\circ$
 $d_{\min} = 0.3207\lambda$

$$Z_R = 25 - j 100 \Omega \quad (Z_0 = 50 \Omega)$$

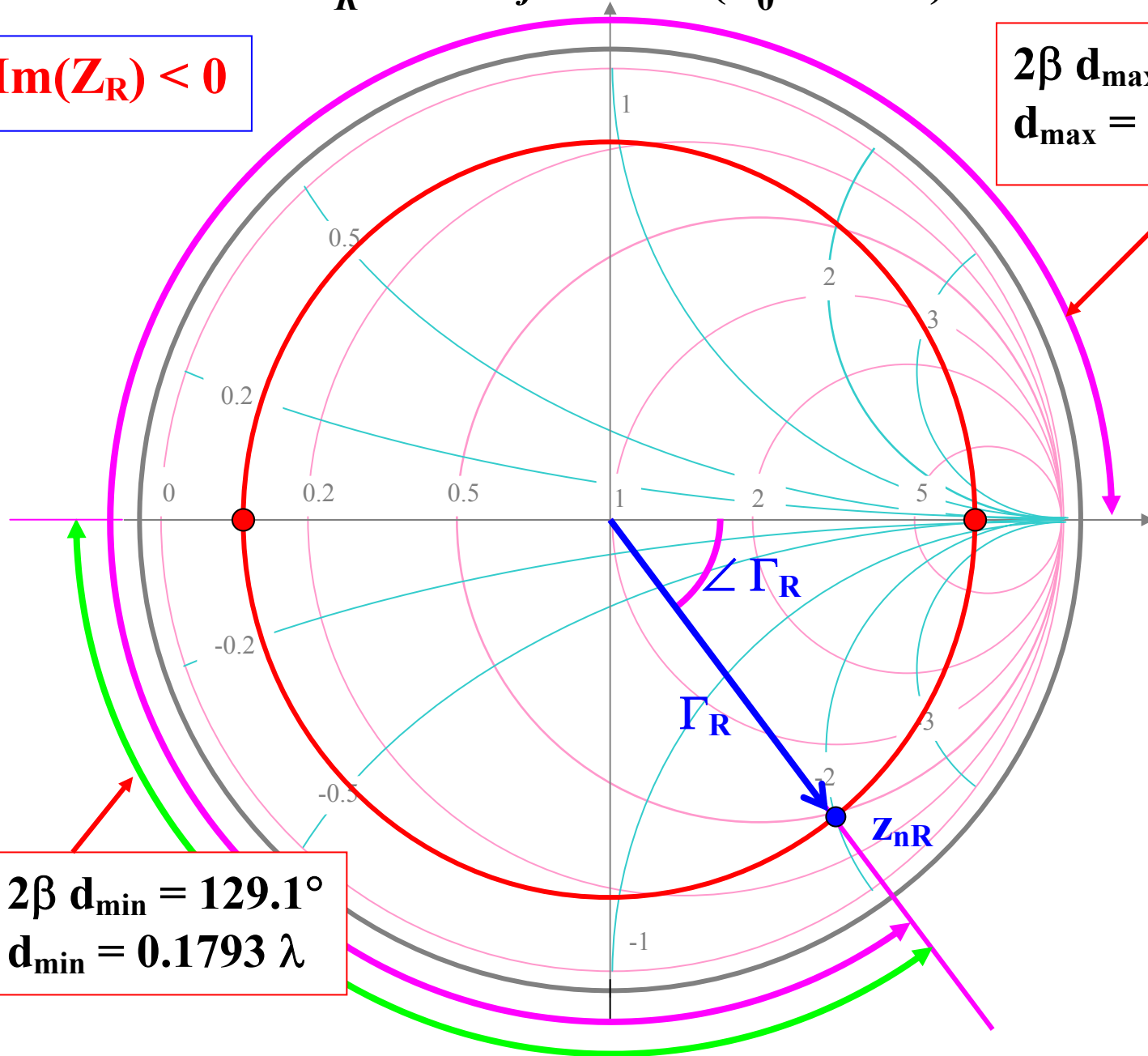
$$\text{Im}(Z_R) < 0$$

$$2\beta d_{\max} = 309.1^\circ$$

$$d_{\max} = 0.4293 \lambda$$

$$2\beta d_{\min} = 129.1^\circ$$

$$d_{\min} = 0.1793 \lambda$$



Given Γ_R and $Z_R \Rightarrow$ Find the Voltage Standing Wave Ratio (VSWR)

The Voltage standing Wave Ratio or **VSWR** is defined as

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}$$

The **normalized impedance** at a **maximum location** of the standing wave pattern is given by

$$z_n(d_{\max}) = \frac{1 + \Gamma(d_{\max})}{1 - \Gamma(d_{\max})} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|} = VSWR!!!$$

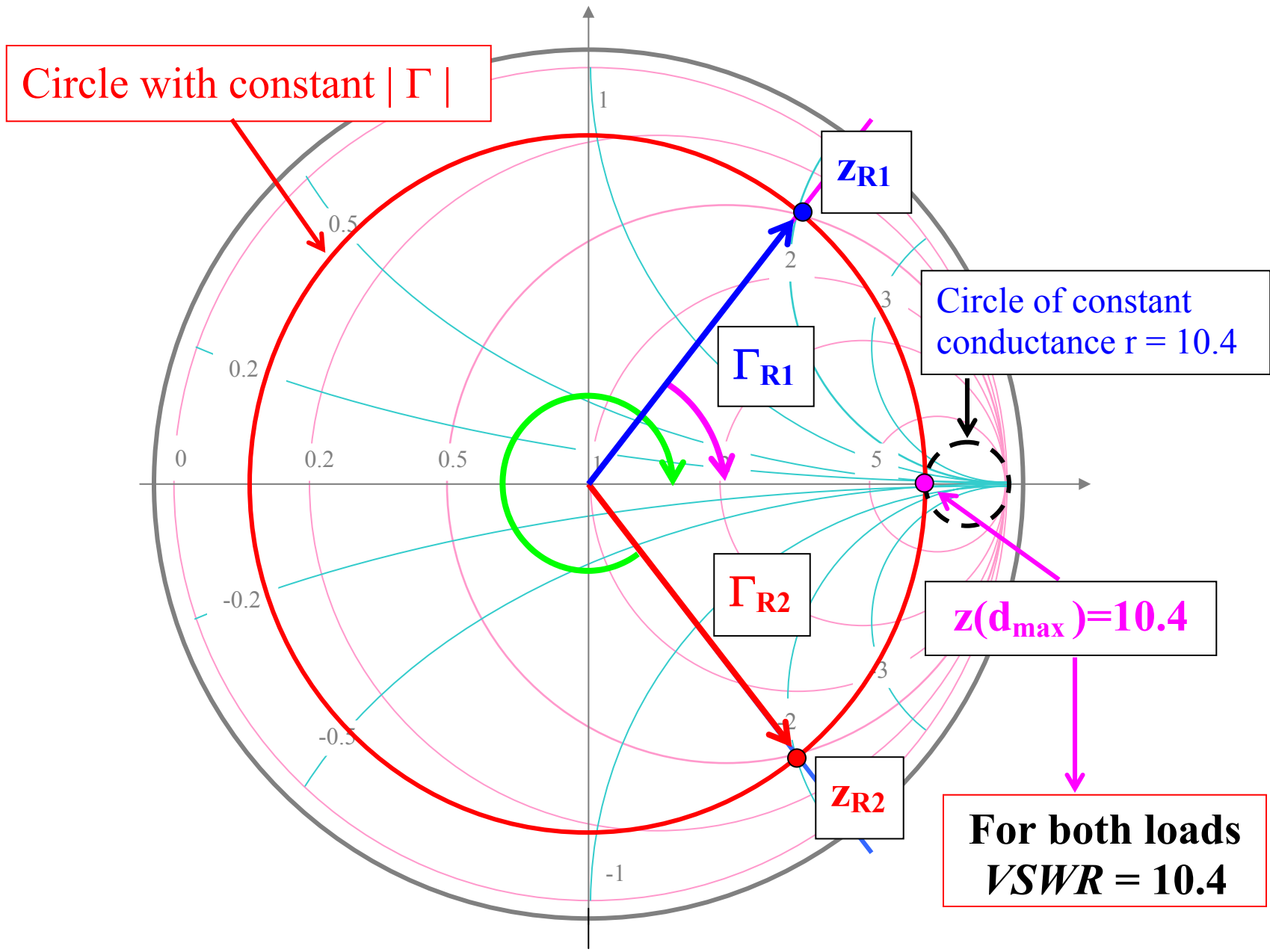
This quantity is always **real** and ≥ 1 . The VSWR is simply obtained on the Smith chart, by reading the value of the (real) normalized impedance, at the location d_{\max} where Γ is **real** and **positive**.

The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.
3. Find the intersection of this circle with the real positive axis for the reflection coefficient (corresponding to the transmission line location d_{\max}).
4. A circle of **constant normalized resistance** will also intersect this point. Read or interpolate the value of the normalized resistance to determine the $VSWR$.

Example: Find the $VSWR$ for

$$Z_{R1} = 25 + j100 \Omega \ ; \ Z_{R2} = 25 - j100 \Omega \quad (Z_0 = 50 \Omega)$$



Given $Z(d) \iff$ Find $Y(d)$

Note: The normalized impedance and admittance are defined as

$$z_n(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \qquad y_n(d) = \frac{1 - \Gamma(d)}{1 + \Gamma(d)}$$

Since

$$\Gamma\left(d + \frac{\lambda}{4}\right) = -\Gamma(d)$$

$$\Rightarrow z_n\left(d + \frac{\lambda}{4}\right) = \frac{1 + \Gamma\left(d + \frac{\lambda}{4}\right)}{1 - \Gamma\left(d + \frac{\lambda}{4}\right)} = \frac{1 - \Gamma(d)}{1 + \Gamma(d)} = y_n(d)$$

Keep in mind that the equality

$$z_n \left(d + \frac{\lambda}{4} \right) = y_n (d)$$

is only valid for **normalized** impedance and admittance. The **actual** values are given by

$$Z \left(d + \frac{\lambda}{4} \right) = Z_0 \cdot z_n \left(d + \frac{\lambda}{4} \right)$$

$$Y(d) = Y_0 \cdot y_n(d) = \frac{y_n(d)}{Z_0}$$

where $Y_0 = 1 / Z_0$ is the **characteristic admittance** of the transmission line.

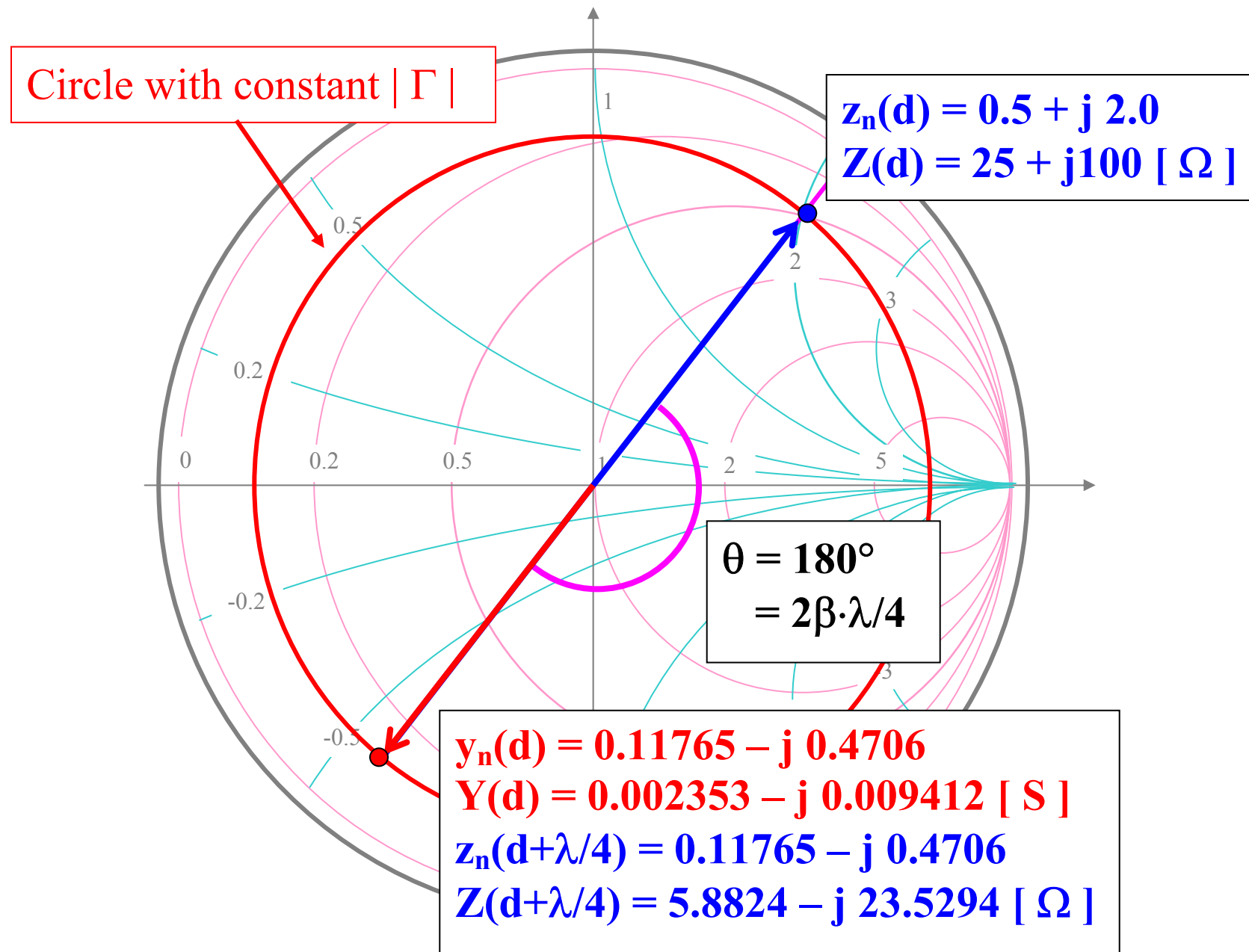
The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.
3. The **normalized admittance** is located at a point on the circle of constant $|\Gamma|$ which is diametrically opposite to the **normalized impedance**.

Example: Given

$$Z_R = 25 + j 100 \Omega \quad \text{with} \quad Z_0 = 50 \Omega$$

find Y_R .



The Smith chart can be used for line admittances, by shifting the space reference to the admittance location. After that, one can move on the chart just reading the numerical values as representing admittances.

Let's review the **impedance-admittance** terminology:

Impedance = Resistance + j Reactance

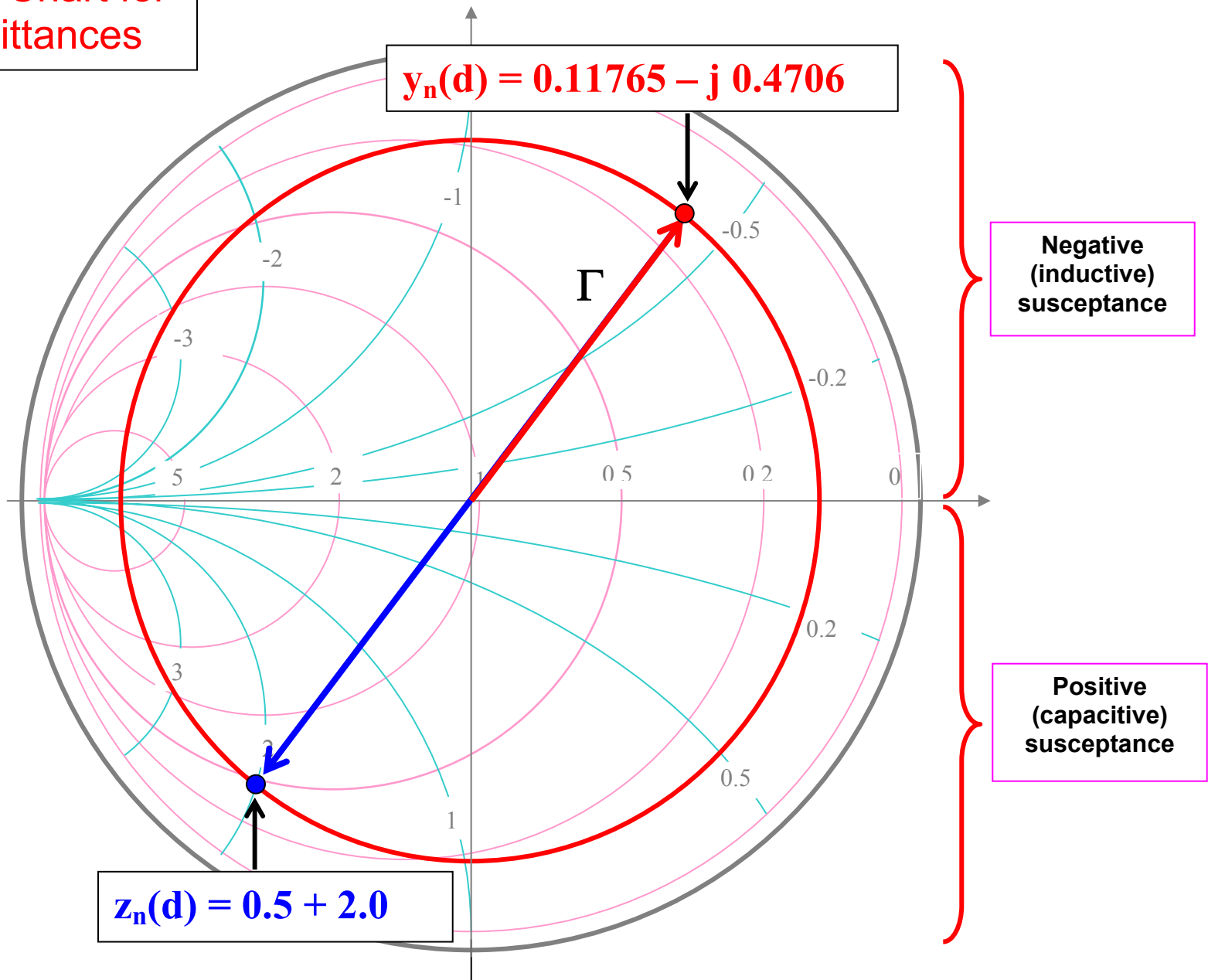
$$Z = R + jX$$

Admittance = Conductance + j Susceptance

$$Y = G + jB$$

On the **impedance** chart, the correct **reflection coefficient** is always represented by the vector corresponding to the **normalized impedance**. Charts specifically prepared for **admittances** are modified to give the correct reflection coefficient in correspondence of admittance.

Smith Chart for Admittances



Since related **impedance** and **admittance** are on opposite sides of the same Smith chart, the imaginary parts always have different sign.

Therefore, a **positive (inductive) reactance** corresponds to a **negative (inductive) susceptance**, while a **negative (capacitive) reactance** corresponds to a **positive (capacitive) susceptance**.

Analytically, the normalized impedance and admittance are related as

$$z_n = r + jx \qquad y_n = g + jb = \frac{1}{r + jx}$$

$$y_n = \frac{r - jx}{(r + jx)(r - jx)} = \frac{r - jx}{r^2 + x^2}$$

$$\Rightarrow \qquad g = \frac{r}{r^2 + x^2} \qquad b = -\frac{x}{r^2 + x^2}$$