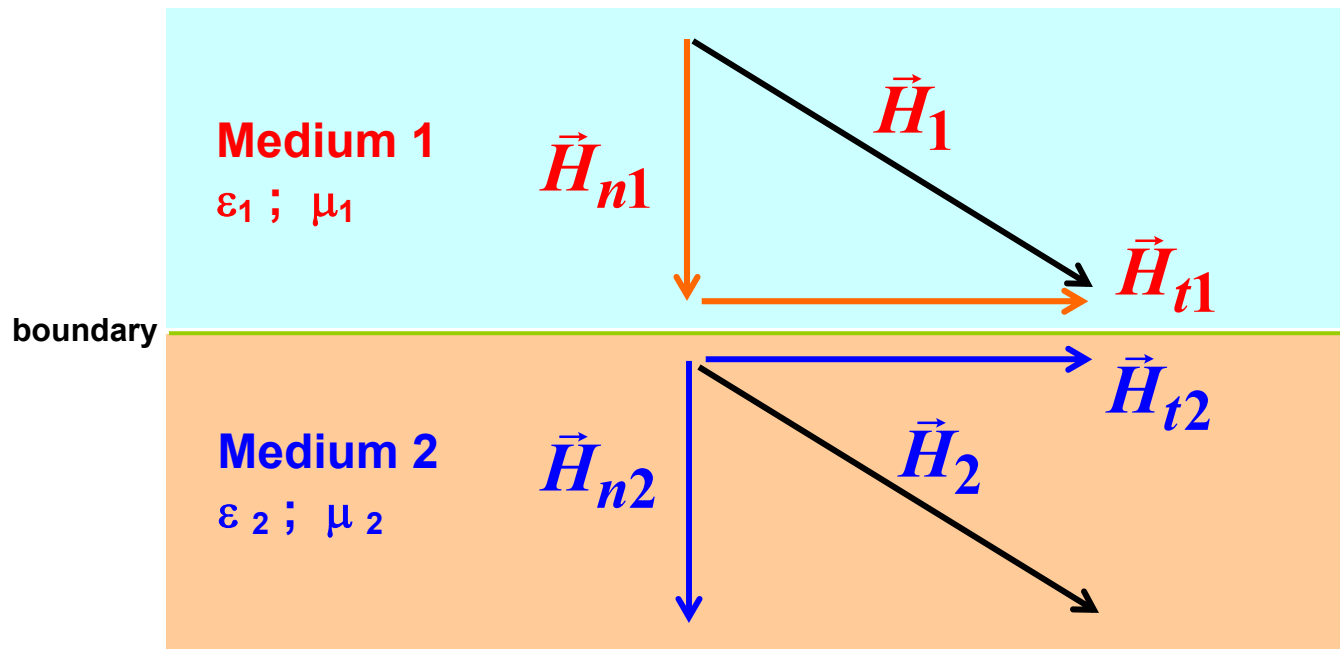
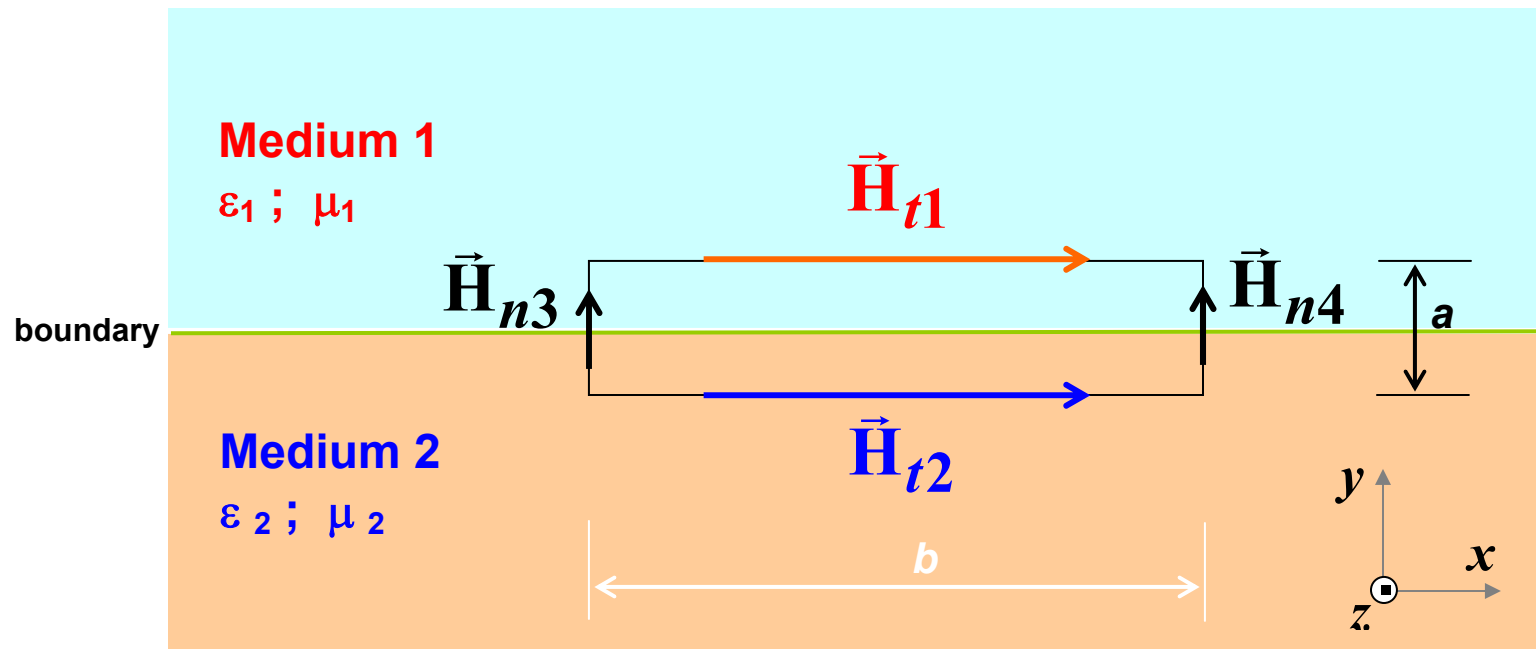


Review of Boundary Conditions

Consider an electromagnetic field at the **boundary** between two materials with different properties. The **tangent** and the **normal** component of the fields must be examined separately, in order to understand the effects of the boundary.



Tangential Magnetic Field



Ampère's law for the boundary region in the figure can be written as

$$\nabla \times \vec{H} \Rightarrow \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + j\omega \epsilon E_z$$

In terms of finite differences approximation for the derivatives

$$\frac{\mathbf{H}_{n4} - \mathbf{H}_{n3}}{b} - \frac{\mathbf{H}_{t1} - \mathbf{H}_{t2}}{a} = \mathbf{J}_z + j\omega \varepsilon \mathbf{E}_z$$

If one lets the boundary region shrink, with a going to zero **faster** than b ,

$$\mathbf{H}_{t2} - \mathbf{H}_{t1} = \lim_{a \rightarrow 0} \left(\mathbf{J}_z a + j\omega \varepsilon \mathbf{E}_z a + a \frac{\mathbf{H}_{n3} - \mathbf{H}_{n4}}{b} \right)$$

for materials with finite conductivity

$$\Rightarrow \mathbf{H}_{t2} - \mathbf{H}_{t1} = \mathbf{0} \quad \text{Tangential components are conserved}$$

for perfect conductors

$$\Rightarrow \mathbf{H}_{t2} - \mathbf{H}_{t1} = \lim_{a \rightarrow 0} (\mathbf{J}_z a) = \mathbf{J}_s \quad (\text{surface current})$$

For a **general** boundary geometry

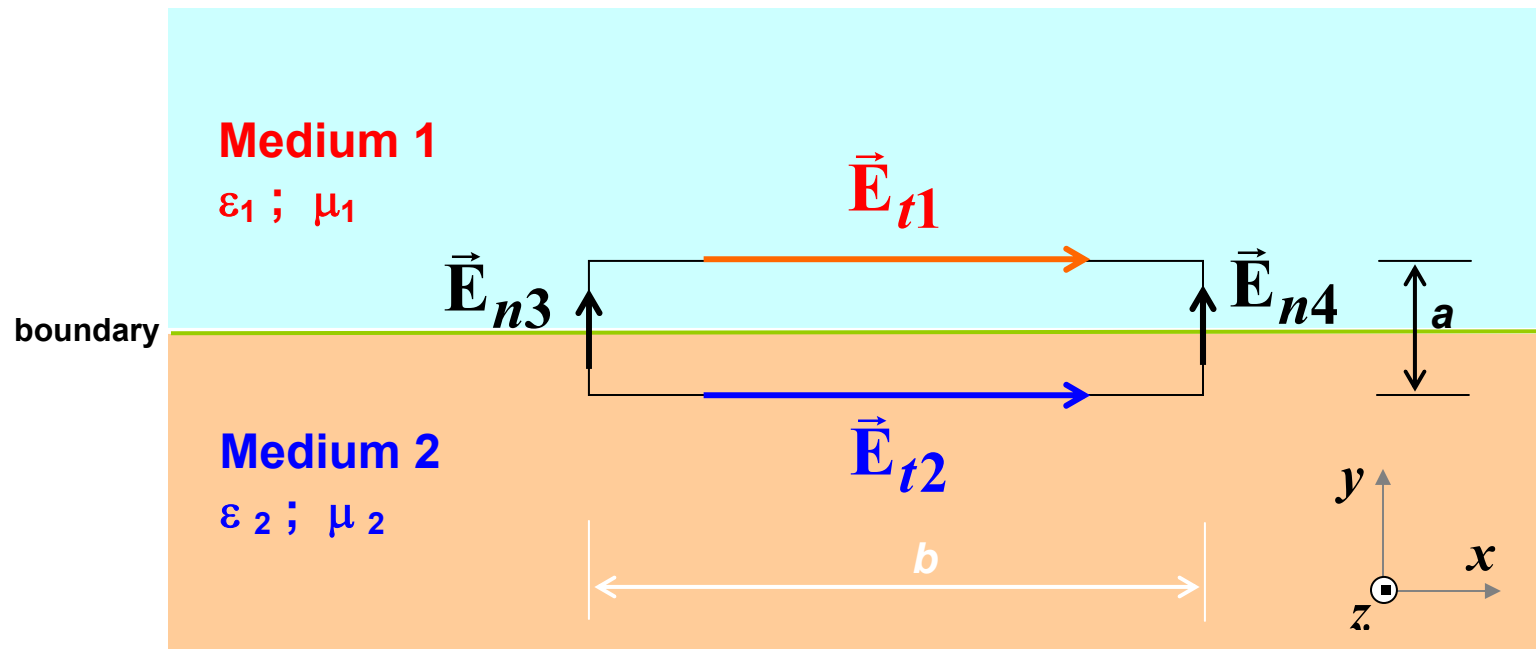
$$\hat{n} \times (\vec{H}_{t1} - \vec{H}_{t2}) = \vec{J}_s$$

\hat{n} = **unit vector normal to the surface**

In the case of a **perfect conductor**, the electromagnetic fields go immediately to zero inside the material, because the **conductivity is infinite** and attenuates instantly the fields. The surface current is confined to an **infinitesimally thin “skin”**, and it accounts for the discontinuity of the tangential magnetic field, which becomes immediately zero inside the perfect conductor.

For a **real medium**, with **finite conductivity**, the fields can penetrate over a certain distance, and there is a current distributed on a thin, but not infinitesimal, skin layer. The tangential field components on the two sides of the interface are the same. **Nonetheless, the perfect conductor is often a good approximation for a real metal.**

Tangential Electric Field



Faraday's law for the same boundary region can be written as

$$\nabla \times \vec{E} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = j\omega \mu H_z$$

In terms of finite differences approximation for the derivatives

$$\frac{\mathbf{E}_{n4} - \mathbf{E}_{n3}}{b} - \frac{\mathbf{E}_{t1} - \mathbf{E}_{t2}}{a} = j\omega\mu \mathbf{H}_z$$

If one lets the boundary region shrink, with a going to zero **faster** than b ,

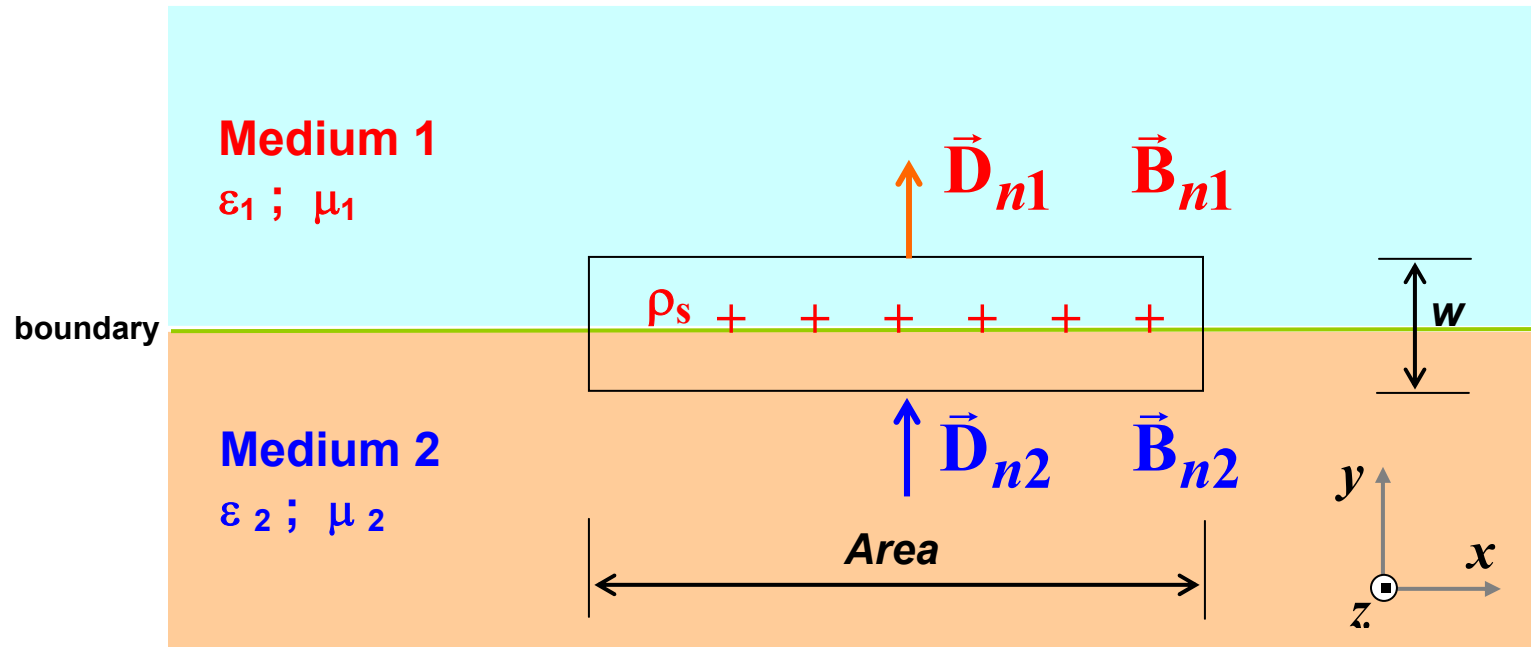
$$\mathbf{E}_{t2} - \mathbf{E}_{t1} = \lim_{a \rightarrow 0} \left(j\omega\mu \mathbf{H}_z a + a \frac{\mathbf{E}_{n3} - \mathbf{E}_{n4}}{b} \right)$$

$$\Rightarrow \mathbf{E}_{t2} - \mathbf{E}_{t1} = \mathbf{0} \quad \text{Tangential components are conserved}$$

For a **general** boundary geometry

$$\hat{n} \times (\vec{\mathbf{E}}_{t1} - \vec{\mathbf{E}}_{t2}) = \mathbf{0}$$

Normal components



Consider a small box that encloses a certain area of the interface with

$$\rho_s = \text{interface charge density}$$

Integrate the divergence of the fields over the volume of the box:

$$\iiint_{\text{Volume}} \nabla \cdot \vec{\mathbf{D}} \, d\vec{\mathbf{r}} = \iiint_{\text{Volume}} \rho \, d\vec{\mathbf{r}}$$

Divergence \Downarrow theorem

$$\oiint_{\text{Surface}} \vec{\mathbf{D}} \cdot \hat{\mathbf{n}} \, d\vec{\mathbf{s}} = \text{Flux of } \vec{\mathbf{D}} \text{ out of the box}$$

$$\iiint_{\text{Volume}} \nabla \cdot \vec{\mathbf{B}} \, d\vec{\mathbf{r}} = 0$$

Divergence \Downarrow theorem

$$\oiint_{\text{Surface}} \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} \, d\vec{\mathbf{s}} = \text{Flux of } \vec{\mathbf{B}} \text{ out of the box}$$

If the **thickness** of the box tends to **zero** and the **charge** density is assumed to be **uniform** over the area, we have the following fluxes

$$\begin{aligned} \vec{\mathbf{D}}\text{-Flux out of box} &= \textit{Area} \cdot (\mathbf{D}_{1n} - \mathbf{D}_{2n}) = \\ &= \text{Total interface charge} = \textit{Area} \cdot \rho_s \end{aligned}$$

$$\vec{\mathbf{B}}\text{-Flux out of box} = \textit{Area} \cdot (\mathbf{B}_{1n} - \mathbf{B}_{2n}) = 0$$

The resulting **boundary conditions** are

$$\mathbf{D}_{1n} - \mathbf{D}_{2n} = \rho_s \qquad \mathbf{B}_{1n} - \mathbf{B}_{2n} = 0$$

The **discontinuity** in the normal component of the displacement field **D** is equal to the density of surface charge.

The **normal components** of the magnetic induction field **B** are **continuous** across the interface.

For isotropic and uniform values of ϵ and μ in the two media

$$\begin{aligned}\vec{\mathbf{D}}_{n1} - \vec{\mathbf{D}}_{n2} &= \epsilon_1 \vec{\mathbf{E}}_{n1} - \epsilon_2 \vec{\mathbf{E}}_{n2} = \rho_s \\ \vec{\mathbf{B}}_{n1} - \vec{\mathbf{B}}_{n2} &= \mu_1 \vec{\mathbf{H}}_{n1} - \mu_2 \vec{\mathbf{H}}_{n2} = 0\end{aligned}$$

Even when the interface charge is zero, the normal components of the electric field are discontinuous at the interface, if there is a change of dielectric constant .

The normal components of the magnetic field have a similar discontinuity at the interface due to the change in the magnetic permeability. **In many practical situations, the two media may have the same permeability as vacuum, μ_0 , and in such cases the normal component of the magnetic field is conserved across the interface.**

SUMMARY

If medium 2 is perfect conductor

$$\begin{array}{c} \vec{H}_{t1} \rightarrow \\ \hline \vec{H}_{t2} \rightarrow \end{array} \quad \begin{array}{l} \epsilon_1, \mu_1 \\ \epsilon_2, \mu_2 \end{array}$$

$$\vec{H}_{t1} = \vec{H}_{t2}$$

$$\hat{n} \times \vec{H}_{t1} = \vec{J}_s$$

$$\vec{H}_{t2} = 0$$

$$\begin{array}{c} \vec{E}_{t1} \rightarrow \\ \hline \vec{E}_{t2} \rightarrow \end{array} \quad \begin{array}{l} \epsilon_1, \mu_1 \\ \epsilon_2, \mu_2 \end{array}$$

$$\vec{E}_{t1} = \vec{E}_{t2}$$

$$\vec{E}_{t1} = 0$$

$$\vec{E}_{t2} = 0$$

$$\begin{array}{c} \downarrow \vec{H}_{n1} \\ \hline \downarrow \vec{H}_{n2} \end{array} \quad \begin{array}{l} \epsilon_1, \mu_1 \\ \epsilon_2, \mu_2 \end{array}$$

$$\mu_1 \vec{H}_{n1} = \mu_2 \vec{H}_{n2}$$

$$\vec{H}_{n1} = 0$$

$$\vec{H}_{n2} = 0$$

$$\begin{array}{c} \downarrow \vec{E}_{n1} \\ \hline \downarrow \vec{E}_{n2} \end{array} \quad \begin{array}{l} \epsilon_1, \mu_1 \\ \epsilon_2, \mu_2 \end{array}$$

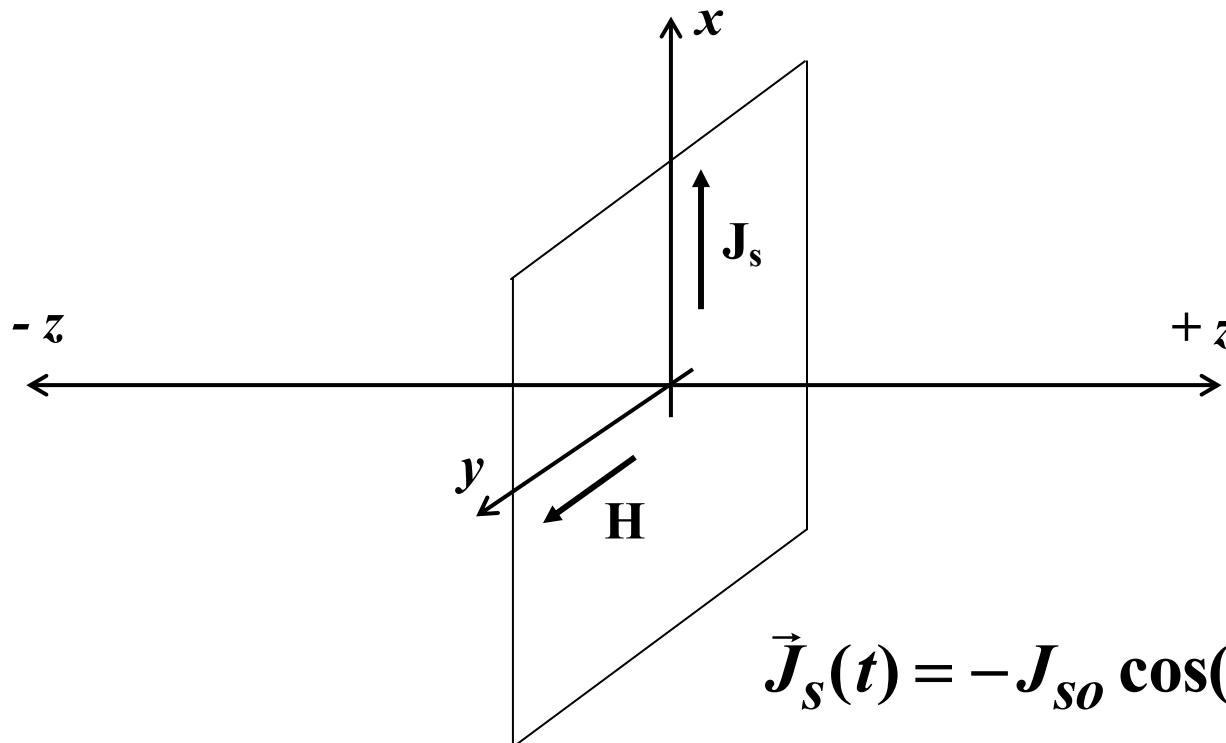
$$\epsilon_1 \vec{E}_{n1} = \epsilon_2 \vec{E}_{n2} + \rho_s$$

$$\vec{E}_{n1} = \rho_s / \epsilon_1$$

$$\vec{E}_{n2} = 0$$

Examples:

An infinite current sheet generates a plane wave (free space on both sides)



$$\vec{\mathbf{J}}_s(t) = -J_{s0} \cos(\omega t) \hat{\mathbf{i}}_x$$

$$\text{Phasor } \vec{\mathbf{J}}_s = -J_{s0} \hat{\mathbf{i}}_x$$

The E.M. field is transmitted on both sides of the infinitesimally thin sheet of current.

BOUNDARY CONDITIONS

$$\hat{n} \times (\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2}) = \vec{\mathbf{J}}_s$$

$$\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2} = J_{so} \hat{i}_x$$

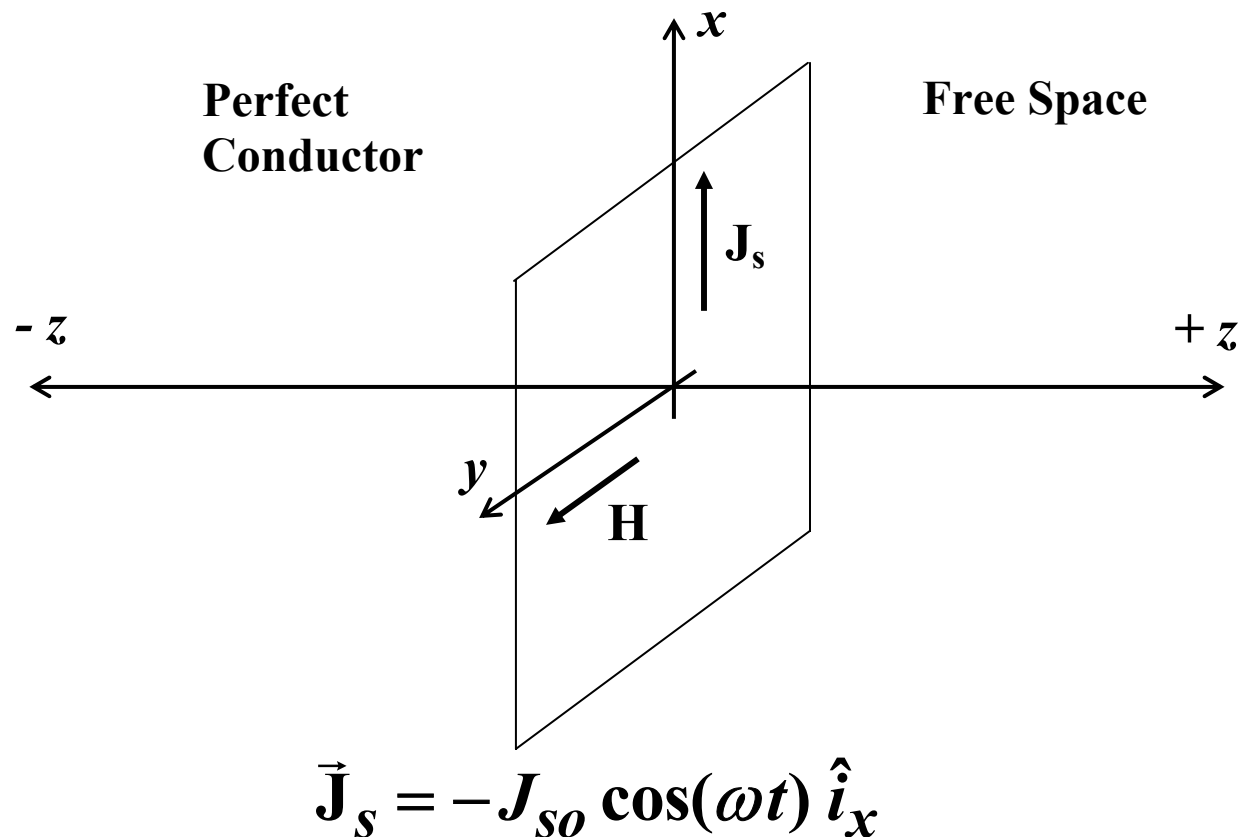
$$\vec{\mathbf{E}}_{t1} = \vec{\mathbf{E}}_{t2}$$

$$|\vec{\mathbf{E}}_{t1}| = \eta_0 |\vec{\mathbf{H}}_{t1}|$$

$$\text{Symmetry} \Rightarrow |\vec{\mathbf{H}}_{t1}| = |\vec{\mathbf{H}}_{t2}|$$

$$\mathbf{H}_1 = \frac{J_{so}}{2} \quad \mathbf{H}_2 = -\frac{J_{so}}{2}$$

A semi-infinite perfect conductor medium in contact with free space has uniform surface current and generates a plane wave



The E.M. field is zero inside the perfect conductor. The wave is only transmitted into free space.

BOUNDARY CONDITIONS

$$\hat{n} \times (\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2}) = \vec{\mathbf{J}}_s$$

$$\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2} = \vec{\mathbf{H}}_{t1} - \mathbf{0} = J_{so} \hat{i}_x$$

$$\vec{\mathbf{E}}_{t2} = \mathbf{0}$$

$$\text{Asymmetry} \Rightarrow |\vec{\mathbf{H}}_{t1}| \neq |\vec{\mathbf{H}}_{t2}|$$

$$\mathbf{H}_{t1} = J_{so} \quad \mathbf{H}_{t2} = \mathbf{0}$$