

## Electromagnetic Waves

For **fast-varying phenomena**, the displacement current cannot be neglected, and the full set of Maxwell's equations must be used

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}(t)}{dt}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}(t)}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The two curl equations are analogous to the **coupled** (first order) equations for voltage and current used in transmission lines. The **solutions** of this system of equations are **waves**. In order to obtain **uncoupled** (second order) equations we can operate with the curl once more. Under the assumption of **uniform isotropic medium**:

$$\nabla \times \nabla \times \vec{E}(t) = -\frac{\partial(\nabla \times \vec{B}(t))}{\partial t} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}(t)$$

$$= -\mu \frac{\partial \vec{J}(t)}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2}$$

$$\nabla \times \nabla \times \vec{H}(t) = \nabla \times \vec{J} + \frac{\partial(\nabla \times \vec{D}(t))}{\partial t} = \nabla \times \vec{J} + \varepsilon \frac{\partial}{\partial t} \nabla \times \vec{E}(t)$$

$$= \nabla \times \vec{J} - \varepsilon \mu \frac{\partial^2 \vec{H}(t)}{\partial t^2}$$

From vector calculus, we also have

$$\nabla \times \nabla \times \vec{E}(t) = \nabla \nabla \cdot \vec{E}(t) - \nabla^2 \vec{E}(t)$$

$$\nabla \times \nabla \times \vec{H}(t) = \nabla \nabla \cdot \vec{H}(t) - \nabla^2 \vec{H}(t) = -\nabla^2 \vec{H}(t)$$

$$\frac{1}{\mu} \nabla \cdot \vec{B}(t) = 0$$

Finally, we obtain the **general wave equations**

$$\nabla^2 \vec{E}(t) - \nabla \nabla \cdot \vec{E}(t) - \mu \varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2} = \mu \frac{\partial \vec{J}(t)}{\partial t}$$

$$\nabla^2 \vec{H}(t) - \mu \varepsilon \frac{\partial^2 \vec{H}(t)}{\partial t^2} = -\nabla \times \vec{J}(t)$$

In a region where the wave solution propagates **away** from **charges** and flowing **currents**, the wave equations can be simplified considerably. In such conditions, we have

$$\rho = 0 \quad \Rightarrow \quad \nabla \cdot \vec{E}(t) = \rho / \varepsilon = 0$$
$$\vec{J}(t) = 0$$

and the **wave equations** assume the familiar form

$$\nabla^2 \vec{E}(t) - \mu \varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2} = 0$$
$$\nabla^2 \vec{H}(t) - \mu \varepsilon \frac{\partial^2 \vec{H}(t)}{\partial t^2} = 0$$

When **currents** and **charges** are involved, the wave equations are difficult to solve, because of the terms

$$\nabla(\nabla \cdot \vec{E}(t)) \quad \text{and} \quad \nabla \times \vec{J}(t)$$

It is more practical to have equations for the **electric potential** and for the **magnetic vector potential**, which contain **linear** source terms dependent on charge and current, as shown below.

We saw earlier that the **divergence** of the **magnetic vector potential** must be specified. The simple choice made in magnetostatics of zero divergence is not suitable for time-varying fields. Among the possible choices, it is convenient to adopt the **Lorenz gauge**

Time-varying fields – Lorenz gauge

$$\nabla \cdot \vec{A}(t) = -\mu \epsilon \frac{\partial \phi}{\partial t}$$

Magnetostatics (d.c.)

$$\nabla \cdot \vec{A} = 0$$

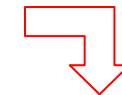
Starting from the definitions

$$\vec{B}(t) = \nabla \times \vec{A} \quad \vec{E}(t) = -\frac{\partial \vec{A}(t)}{\partial t} - \nabla \phi$$

we obtain again the wave equation by applying the curl operation

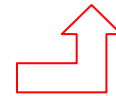
$$\mu \nabla \times \vec{H}(t) = \nabla \times \nabla \times \vec{A}(t) = \nabla \nabla \cdot \vec{A}(t) - \nabla^2 \vec{A}(t) =$$

$$= \mu \vec{J} - \epsilon \mu \frac{\partial^2 \vec{A}(t)}{\partial t^2} - \epsilon \mu \nabla \frac{\partial \phi}{\partial t}$$



$$= \nabla \nabla \cdot \vec{A}(t)$$

With the application of **Lorenz gauge**



$$\nabla^2 \vec{A}(t) - \epsilon \mu \frac{\partial^2 \vec{A}(t)}{\partial t^2} = -\mu \vec{J}(t)$$

For the **electric potential** we have

$$\nabla \cdot \vec{D}(\mathbf{t}) = \rho \quad \Rightarrow \quad \nabla \cdot \vec{E}(\mathbf{t}) = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \vec{E}(\mathbf{t}) = \nabla \cdot \left( -\frac{\partial \vec{A}(\mathbf{t})}{\partial t} - \nabla \phi \right) = \frac{\rho}{\epsilon}$$

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \vec{A}(\mathbf{t}) = \frac{\rho}{\epsilon}$$

After applying the **Lorenz gauge** once more, we arrive at the potential wave equation

$$\nabla^2 \phi - \epsilon \mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}$$

In engineering it is very important to consider **time-harmonic fields** with a sinusoidal time-variation. If we assume a **steady-state** situation (after all **transients** have died out) most physical situations may be investigated by considering one **single frequency** at a time.

This assumption leads to great **simplifications** in the algebra. It is also **realistic**, because in practical electromagnetics applications we often have a dominant frequency (**carrier**) to consider.

The time-harmonic fields have the form

$$\vec{E}(t) = \vec{E}_0 \cos(\omega t + \varphi_E) \quad \vec{H}(t) = \vec{H}_0 \cos(\omega t + \varphi_H)$$

We can use the **complex phasor representation**

$$\vec{E}(t) = \text{Re} \left\{ \vec{E}_0 e^{j\varphi_E} e^{j\omega t} \right\} \quad \vec{H}(t) = \text{Re} \left\{ \vec{H}_0 e^{j\varphi_H} e^{j\omega t} \right\}$$

We define

$$\vec{\mathbf{E}} = \vec{E}_0 e^{j\varphi_E} = \text{phasor of } \vec{E}(t)$$

$$\vec{\mathbf{H}} = \vec{H}_0 e^{j\varphi_H} = \text{phasor of } \vec{H}(t)$$

**Maxwell's equations** can be rewritten for **phasors**, with the **time-derivatives** transformed into **linear terms**

$$j\omega \vec{\mathbf{E}} = \text{phasor of } \frac{\partial \vec{E}(t)}{\partial t}$$

$$-\omega^2 \vec{\mathbf{E}} = \text{phasor of } \frac{\partial^2 \vec{E}(t)}{\partial t^2}$$

In **phasor form**, Maxwell's equations become

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu \vec{\mathbf{H}}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\varepsilon \vec{\mathbf{E}}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}$$

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$$

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$$

where all electromagnetic quantities are **phasors** and **functions** only of **space coordinates**.

Let's consider first **vacuum** as a medium. The wave equations for **phasors** become **Helmholtz equations**

$$\nabla^2 \vec{E} + \omega^2 \mu_0 \epsilon_0 \vec{E} = 0$$

$$\nabla^2 \vec{H} + \omega^2 \mu_0 \epsilon_0 \vec{H} = 0$$

The **general solutions** for these differential equations are **waves** moving in 3-D space. Note, once again, that the two equations are **uncoupled**.

This means that each equation contains all the necessary information for the total electromagnetic field and one only needs to **solve** the equation for **one field** to completely specify the problem. The other field is obtained with a curl operation by invoking one of the original Maxwell equations.

At this stage we **assume that a wave exists**, and we do not yet concern ourselves with the way the wave is generated. So, for the sake of understanding wave behavior, we can **restrict** the Helmholtz equations to a **simple case**:

- We assume that the wave solution has an **electric field** which is **uniform** on the  $\{x, y\}$ -plane and has a reference positive orientation along the  $x$ -direction. Then, we verify that this is a reasonable choice corresponding to an actual solution of the Helmholtz wave equations. We recall that the **Laplacian** of a **scalar** is a scalar

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and that the **Laplacian** of a **vector** is a vector

$$\nabla^2 \vec{\mathbf{E}} = \hat{i}_x \nabla^2 \mathbf{E}_x + \hat{i}_y \nabla^2 \mathbf{E}_y + \hat{i}_z \nabla^2 \mathbf{E}_z$$

The **Helmholtz equation** becomes:

$$\nabla^2 \vec{\mathbf{E}} + \omega^2 \mu_0 \varepsilon_0 \vec{\mathbf{E}} = \frac{\partial^2 \mathbf{E}_x}{\partial z^2} \hat{i}_x + \omega^2 \mu_0 \varepsilon_0 (\mathbf{E}_x \hat{i}_x) = \mathbf{0}$$

Only the **x-component** of the electric field exists (due to the chosen orientation) and only the **z-derivative** exists, because the **field** is **uniform** on the  $\{x, y\}$ -plane.

We have now a **one-dimensional** wave propagation problem described by the **scalar** differential equation

$$\frac{\partial^2 \mathbf{E}_x}{\partial z^2} + \omega^2 \mu_0 \varepsilon_0 \mathbf{E}_x = \mathbf{0}$$

This equation has a well known **general solution**

$$A \exp(-j\beta z) + B \exp(j\beta z)$$

where the **propagation constant** is

$$\beta = \omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c}$$

The wave that we have assumed is a **plane wave** and we have verified that it is a **solution** of Helmholtz equation. The general solution above has two possible components

$A \exp(-j\beta z)$                       Forward wave, moving along positive  $z$

$B \exp(j\beta z)$                       Backward wave, moving along negative  $z$

**For the simple wave orientation chosen here, the problem is mathematically identical to the one solved earlier for voltage propagation in a homogeneous transmission line.**

If a specific electromagnetic wave is established in an infinite homogeneous medium, moving for instance along the positive direction, only the forward wave should be considered.

A reflected wave exists when a discontinuity takes place along the path of the forward wave (that is, the material medium changes properties, either abruptly or gradually).

We can also assume that the **amplitude** of the forward plane wave solution is given and that it is in general a **complex** constant fixed by the conditions that generated the wave

$$A = E_0 e^{j\phi}$$

We can write at last the **phasor electric field** describing a simple **forward plane wave solution** of Helmholtz equation as:

$$\vec{E}_x(z) = E_0 e^{j\phi} e^{-j\beta z} \hat{i}_x$$

The corresponding **time-dependent** field is obtained by applying the **inverse** phasor transformation

$$\begin{aligned}\vec{E}_x(z, t) &= \text{Re} \left\{ \mathbf{E}_x(z) e^{j\omega t} \hat{i}_x \right\} = \text{Re} \left\{ E_0 e^{j\varphi} e^{-j\beta z} e^{j\omega t} \hat{i}_x \right\} \\ &= E_0 \cos(\omega t - \beta z + \varphi) \hat{i}_x\end{aligned}$$

The **phasor magnetic field** is obtained directly from the Maxwell equation for the electric field curl

$$\begin{aligned}\nabla \times \vec{E} &= \nabla \times \left( E_0 e^{j\varphi} e^{-j\beta z} \hat{i}_x \right) = -j\omega\mu_0 \vec{H} \\ \vec{H} &= - \frac{\nabla \times \left( E_0 e^{j\varphi} e^{-j\beta z} \hat{i}_x \right)}{j\omega\mu_0}\end{aligned}$$

We then develop the curl as

$$\begin{aligned} \nabla \times \left( \underbrace{E_0 e^{j\varphi} e^{-j\beta z}}_{\mathbf{E}_x(z)} \hat{i}_x \right) &= \det \begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{E}_x(z) & \mathbf{0} & \mathbf{0} \end{bmatrix} = \\ &= \frac{\partial \left( E_0 e^{j\varphi} e^{-j\beta z} \right)}{\partial z} \hat{i}_y - \underbrace{\frac{\partial \left( E_0 e^{j\varphi} e^{-j\beta z} \right)}{\partial y}}_{\substack{\uparrow \\ = 0}} \hat{i}_z = \\ &= -j\beta E_0 e^{j\varphi} e^{-j\beta z} \hat{i}_y \end{aligned}$$

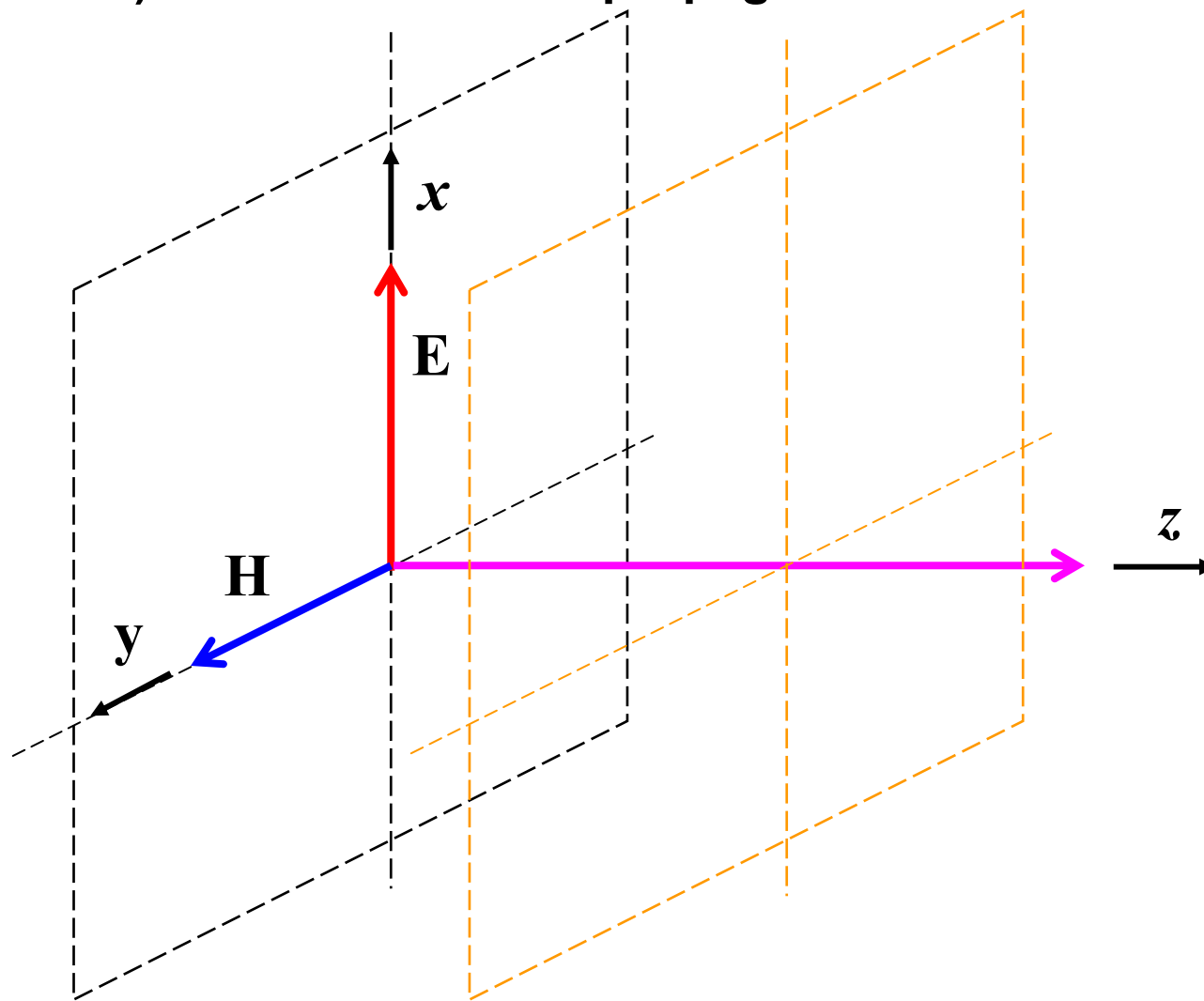
The final result for the **phasor magnetic field** is

$$\begin{aligned}
 \vec{H}_y(z) &= -\frac{-j\beta E_0 e^{j\varphi} e^{-j\beta z}}{j\omega\mu} \hat{i}_y = \\
 &= \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0} E_0 e^{j\varphi} e^{-j\beta z} \hat{i}_y = \\
 &= \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 e^{j\varphi} e^{-j\beta z} \hat{i}_y = \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{E}_x(z) \hat{i}_y
 \end{aligned}$$

We define

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 \approx 377 \Omega = \text{Intrinsic impedance of vacuum}$$

We have found that the **fields** of the **electromagnetic wave** are **perpendicular** to each other, and that they are also perpendicular (or **transverse**) to the direction of propagation.



**Electromagnetic power** flows with the wave along the direction of propagation and it is also constant on the phase-planes. The **power density** is described by the time-dependent **Poynting vector**

$$\vec{P}(t) = \vec{E}(t) \times \vec{H}(t)$$

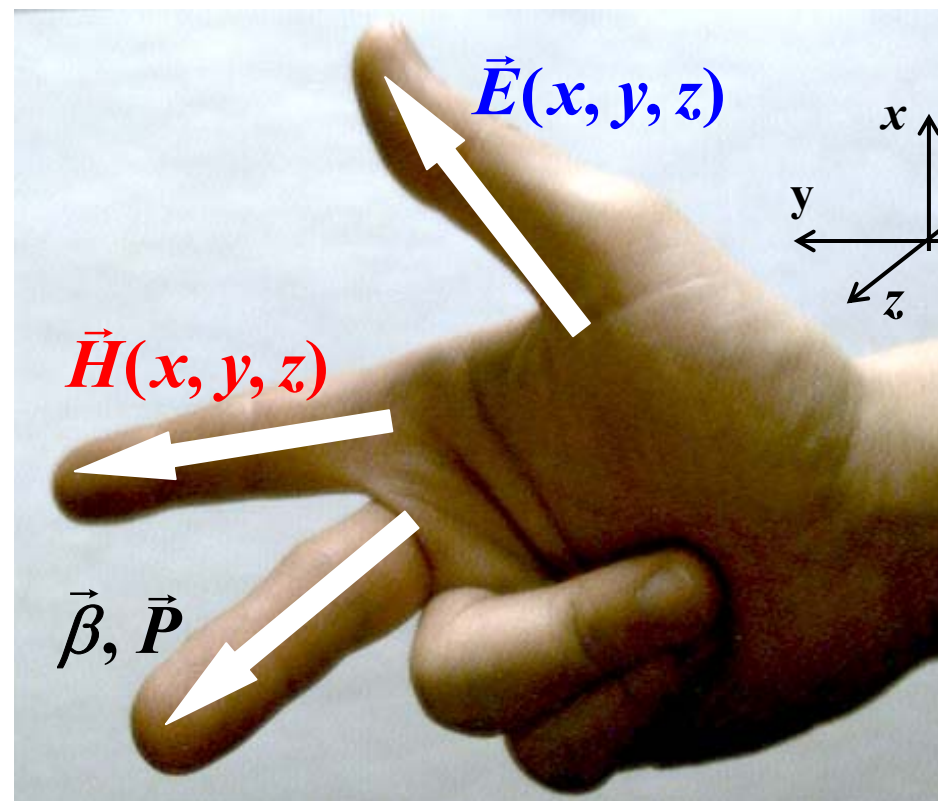
The Poynting vector is perpendicular to both field components, and is parallel to the direction of wave propagation.

When the wave propagates on a general direction, which does not coincide with one of the cartesian axes, the **propagation constant** must be considered to be a vector with amplitude

$$|\vec{\beta}| = \omega \sqrt{\mu \epsilon}$$

and direction **parallel** to the **Poynting vector**.

The condition of mutual orthogonality between the field components and the Poynting vector is **general** and it applies to any plane wave with arbitrary direction of propagation. The mutual orientation chosen for the reference directions of the fields **follows the right hand rule.**



So far, we have just verified that electromagnetic plane waves are possible solutions of the Maxwell equations for time-varying fields. **One may wonder at this point if plane waves have practical physical relevance.**

First of all, we should notice that **plane waves** are **mathematically analogous** to the exponential basis functions used in **Fourier analysis**. This means that a general wave, with more than one frequency component, can always be decomposed in terms of plane waves.

- For **periodic signals**, we have a discrete set of waves which are harmonics of the fundamental frequency (**analogy with Fourier series**).
- For **general signals**, we must consider a continuum of frequencies in order to decompose in terms of elementary plane waves (**analogy with Fourier transform**).

From a **physical point of view**, however, the properties of a plane wave may be somewhat puzzling.

Assume that a **steady-state** plane wave is established in an ideal infinite **homogeneous** medium. On any plane perpendicular to the direction of propagation (phase-planes), the electric and magnetic fields have **uniform** magnitude and phase.

The **electromagnetic power**, flowing with a phase-plane of the wave, is obtained by integrating the **Poynting vector**, which is also uniform on each phase-plane. **For a plane where the Poynting vector is non-zero, the total power carried by the wave is infinite**

$$\int_{plane} \vec{P}(t) = \int_{plane} \vec{E}(t) \times \vec{H}(t) \rightarrow \infty$$

**In many practical cases, we approximate an actual wave with a plane wave on a limited region of space, thus considering an appropriate finite power.**