

Review: Time – Varying Fields

In the **dynamics** case, we can distinguish between two regimes:

Low Frequency (Slowly-Varying Fields) – The displacement current is negligible in the Maxwell's equations, since

$$\left| \frac{\partial \vec{D}(t)}{\partial t} \right| \ll |\vec{J}(t)|$$

High Frequency (Fast-Varying Fields) – The general set of Maxwell's equations must be considered, with no approximations.

In the **low frequency** regime we use the complete set of Maxwell's equations, but the **displacement current** is omitted

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{dt}$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The concept of **low frequency** and **slowly-varying phenomena** is **relative** to the situation at hand. Any **disturbance** (time-variation) of the electromagnetic field propagates at the speed of light. If a length **L** is the maximum dimension of the system under study, the maximum propagation time for a disturbance is

$$\text{Maximum Propagation Time} \Rightarrow t_d = \frac{\text{Maximum Length}}{\text{Phase velocity of light}}$$

We can assume slow-varying fields if the currents are practically constant during this time period.

For **sinusoidal currents**, with a period of oscillation **T**, we have

$$\text{Period} \Rightarrow T = \frac{1}{\text{Frequency}} = \frac{\text{Wavelength}}{v_p} \gg t_d \quad \text{and} \quad L \ll \lambda$$

The **electric potential** is now by itself **insufficient** to completely describe the time-varying **electric field**, because there is also a direct dependence on the magnetic field variations. By recalling the definition of **magnetic vector potential**, we can derive a relationship between **electric field** and **electric potential**

Time-Varying Fields

$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A}(t)$$

$$\Rightarrow \nabla \times \left(\vec{E}(t) + \frac{\partial \vec{A}(t)}{\partial t} \right) = \mathbf{0}$$

$$\vec{E}(t) + \frac{\partial \vec{A}(t)}{\partial t} = -\nabla \phi(t)$$

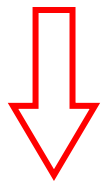
Statics

$$\nabla \times \vec{E} = \mathbf{0}$$

$$\vec{E} = -\nabla \phi$$

We can also obtain an integral relation between **electric field** and **magnetic flux**, by integrating the curl of the electric field over a surface

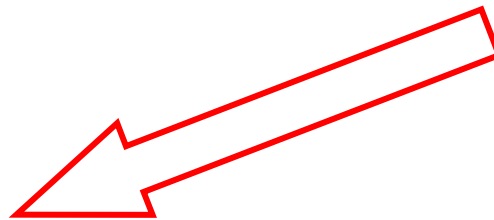
$$\iint_S \nabla \times \vec{E}(t) \cdot d\vec{S} = \iint_S -\frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{S} = -\frac{\partial}{\partial t} \iint_S \vec{B}(t) \cdot d\vec{S}$$



Stoke's Theorem

Magnetic Flux $\Phi(t)$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi(t)}{\partial t}$$



In the electrostatic case, we do not need to distinguish between **voltage** and **potential difference**. The **voltage** between two points is always defined as

$$V_{ba} = -\int_a^b \vec{E} \cdot d\vec{l} = -e.m.f.$$

but in terms of potential ϕ we have

Time-Varying Fields
$V_{ba}(t) = \int_a^b \left(\nabla\phi + \frac{\partial \vec{A}}{\partial t} \right) \cdot d\vec{l}$ $= \phi_b - \phi_a + \frac{\partial}{\partial t} \int_a^b \vec{A}(t) \cdot d\vec{l}$

Statics
$V_{ba} = -\int_a^b \vec{E} \cdot d\vec{l}$ $= \phi_b - \phi_a$

Note that for time-varying fields the line integral of the magnetic vector potential between two given points **depends on** the actual **path of integration**. In general:

$$\int_a^b \vec{A}(t) \cdot d\vec{l} \neq \vec{A}(b, t) - \vec{A}(a, t)$$

Consider now the integral of the electric field along a closed path:

Time-varying fields
$\oint \vec{E}(t) \cdot d\vec{l} \neq \mathbf{0}$

Statics
$\oint \vec{E}(t) \cdot d\vec{l} = \mathbf{0}$

The closed path could be a metallic wire which confines the current due to moving electric charge.

The line integral of the electric field gives the work necessary to move a unit charge along the path of integration, under the influence of time-varying electric and magnetic fields.

For a **closed wire loop** at **rest**, the **work** necessary to move a unit charge **once** around the loop is

$$\begin{aligned}
 W &= \oint \frac{\text{Force}}{\text{Charge}} \cdot d\vec{l} = \oint \vec{E}(t) \cdot d\vec{l} = \int \nabla \times (\vec{E}(t)) \cdot d\vec{S} \\
 &= \int_S -\frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int_S \vec{B}(t) \cdot d\vec{S} \\
 &= -\frac{\partial}{\partial t} \underbrace{\Phi(t)}_{\substack{\text{Magnetic Flux}}}
 \end{aligned}$$

As a more general case, consider a **wire loop in motion**. The complete **Lorentz force** must be considered:

$$\begin{aligned}
 W = e.m.f. &= \oint \frac{\text{Force}}{\text{Charge}} \cdot d\vec{l} = \oint (\vec{E}(t) + \vec{v}(t) \times \vec{B}(t)) \cdot d\vec{l} \\
 &= \int (\nabla \times (\vec{E}(t) + \vec{v}(t) \times \vec{B}(t))) \cdot d\vec{S} \\
 &= \int \left(\underbrace{-\frac{\partial \vec{B}(t)}{\partial t} + \nabla \times (\vec{v}(t) \times \vec{B}(t))}_{\frac{d\vec{B}(t)}{dt}} \right) \cdot d\vec{S} = -\frac{d}{dt} \int \underbrace{\vec{B}(t) \cdot d\vec{S}}_{\text{Flux } \Phi(t)}
 \end{aligned}$$

If the **velocity** of motion is **constant**, note that

$$\begin{aligned}
 \nabla \times (\vec{v} \times \vec{B}(t)) &= \\
 \underbrace{\vec{v} \nabla \cdot \vec{B}}_0 - \underbrace{\vec{B} \nabla \cdot \vec{v}}_0 + \underbrace{(\vec{B} \cdot \nabla) \vec{v}}_0 - (\vec{v} \cdot \nabla) \vec{B} &= -(\vec{v} \cdot \nabla) \vec{B}
 \end{aligned}$$