

Review: Electrostatics and Magnetostatics

In the **static regime**, electromagnetic quantities **do not vary** as a function of **time**. We have two main cases:

ELECTROSTATICS – The electric charges do not change position in time. Therefore, ρ , E and D are constant and there is no magnetic field H , since there is no current density J .

MAGNETOSTATICS – The charge crossing a given cross-section (current) does not vary in time. Therefore, J , H and B are constant. Although charges are moving, the steady current maintains a constant charge density ρ in space and the electric field E is static.

The equations of **electrostatics** are obtained directly from Maxwell's equations, by assuming that $\partial/\partial t$, \mathbf{J} , \mathbf{H} and \mathbf{B} are all zero:

$$\nabla \times \vec{E} = \mathbf{0}$$

$$\nabla \cdot \vec{D} = 0$$

$$\vec{D} = \epsilon \vec{E}$$

The **electrostatic force** is simply

$$\vec{F} = q \vec{E}$$

We also define the **electrostatic potential** ϕ by the relationship

$$\vec{E} = -\nabla\phi$$

The **electrostatic potential** ϕ is a scalar function of space. From vector calculus we know that

$$\nabla \times \nabla \phi = \mathbf{0}$$

This **potential** is very convenient for practical applications because it is a scalar quantity. The potential **automatically satisfies** Maxwell's curl equation for the electric field, since

$$\nabla \times \vec{E} = -\nabla \times \nabla \phi = \mathbf{0}$$

From a **physical** point of view, the electrostatic potential provides an immediate way to express the work W performed by moving a charge from location a to location b :

$$W = -\int_a^b \vec{F} \cdot d\vec{l}$$

Here, l is the coordinate along the path. The **negative sign** indicates that the **work** is done **against** the electrical force.

By introducing the **electrostatic potential**, we obtain

$$W = -q \int_a^b \vec{E} \cdot d\vec{l} = q \int_a^b \nabla \phi \cdot d\vec{l} = q [\phi(b) - \phi(a)] = q \delta\phi$$

NOTE: The line integral of a gradient does not depend on the path of integration but only on the potential at the end points of the path.

The electrostatic potential is measured with respect to an **arbitrary reference value**. We can assume for most problems that a convenient reference is a **zero** potential at an **infinite** distance. In the result above for the electrostatic work, we could set a **zero** potential at the **initial point** of the path, so that $\phi(a)=0$. Either choice of potential reference would give the same potential difference $\delta\phi$.

It is quite convenient to express also the **divergence equation** in terms of the electrostatic potential. If we assume a **uniform** material medium:

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = -\varepsilon \nabla \cdot \nabla \phi = -\varepsilon \nabla^2 \phi = \rho$$

This result yields the well known **Poisson equation**

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon}$$

In the case of $\rho = 0$, we have the classic **Laplace equation**

$$\nabla^2 \phi = 0$$

If the problem involves a **non-uniform medium** with varying dielectric permittivity, a more general form of Poisson equation must be used

$$\nabla \cdot (\epsilon \nabla \phi) = -\rho$$

Another important equation is obtained by integrating the divergence over a certain volume V

$$\int_V \nabla \cdot \vec{D} \, dV = \int_V \rho \, dV$$

Gauss theorem allows us to transform the **volume integral** of the divergence into a **surface integral** of the **flux**

$$\int_V \nabla \cdot \vec{D} \, dV = \int_S \underbrace{\vec{D} \cdot d\vec{S}}_{\text{Component normal to the surface}}$$

Component normal
to the surface

The volume integral of the **charge density** is simply the total charge Q contained inside the volume

$$\int_V \rho \, dV = Q$$

The final result is the **integral form** of Poisson equation, known as **Gauss law**:

$$\int_S \vec{D} \cdot d\vec{S} = Q$$

Most electrostatic problems can be solved by direct application of **Poisson equation** or of **Gauss law**.

Analytical solutions are usually possible only for **simplified** geometries and charge distributions, and **numerical solutions** are necessary for most **general** problems.

The **Gauss law** provides a direct way to determine the **force** between charges. Let's consider a sphere with radius r surrounding a charge Q_1 located at the center. The displacement vector will be **uniform** and **radially** directed, anywhere on the sphere surface, so that

$$\int_S \vec{D} \cdot d\vec{S} = 4\pi r^2 |\vec{D}| = Q_1$$

Assuming a **uniform isotropic medium**, we have a radial electric field with strength

$$|\vec{E}(r)| = \frac{|\vec{D}|}{\epsilon} = \frac{Q_1}{4\pi\epsilon r^2}$$

If a second charge Q_2 is placed at distance r from Q_1 the mutual force has strength

$$|\vec{F}| = Q_2 |\vec{E}(r)| = \frac{Q_1 Q_2}{4\pi\epsilon r^2}$$

The **electrostatic potential** due to a charge Q can be obtained using the previous result for the electrostatic work:

$$\begin{aligned}\phi(b) &= \phi(a) - \int_a^b |\vec{E}| dr = \phi(a) - \int_a^b \frac{Q}{4\pi\epsilon r^2} dr \\ &= \phi(a) + \frac{Q}{4\pi\epsilon} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]\end{aligned}$$

where r indicates the distance of the **observation** point b from the charge location, and a is a reference point. If we chose the reference point $a \rightarrow \infty$, with a reference potential $\phi(a) = 0$, we can express the potential at distance r from the charge Q as

$$\phi(r) = \frac{Q}{4\pi\epsilon r}$$

The **potential** ϕ indicates then the **work** necessary to move an infinitesimal positive probe charge from distance r (point b) to **infinity** (point a) for negative Q , or conversely to move the probe from **infinity** to distance r for positive Q (remember that the work is done against the field). The probe charge should be infinitesimal, not to perturb the potential established by the charge Q .

The work per **unit charge** done **by** the fields to move a probe charge between two points, is usually called **Electromotive Force** (emf). Dimensionally, the emf really represents **work** rather than an actual **force**.

The work per unit charge done **against** the fields represents the **voltage** V_{ba} between the two points, so that:

$$emf = \int_a^b \vec{E} \cdot d\vec{l} = -V_{ba}$$

In **electrostatics**, there is **no difference** between **voltage** and **potential**. To summarize once again, using formulas, we have

Potential at point “ *a* ”

$$\phi_a = -\int_{\infty}^a \vec{E} \cdot d\vec{l}$$

Potential at point “ *b* ”

$$\phi_b = -\int_{\infty}^b \vec{E} \cdot d\vec{l}$$

Potential difference or Voltage between “ *a* ” and “ *b* ”

$$(\phi_b - \phi_a) = V_{ba} = -\int_a^b \vec{E} \cdot d\vec{l} = -e.m.f.$$

In the case of more than one point charge, the **separate potentials** due to each charge can be **added** to obtain the total potential

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon} \sum_i \frac{Q_i}{r_i}$$

If the charge is **distributed** in space with a density $\rho(x, y, z)$, one needs to integrate over the volume as

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(x', y', z')}{r} dV$$

In the **magnetostatic** regime, there are **steady currents** in the system under consideration, which generate magnetic fields (we **ignore** at this point the case of **ferromagnetic** media). The **full set of Maxwell's equations** is considered (setting $\partial/\partial t = 0$)

$$\nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon \vec{E}$$

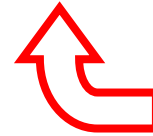
$$\vec{B} = \mu \vec{H}$$

with the complete Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

It is desirable to find also for the **magnetic field** a **potential** function. However, note that such a potential **cannot be a scalar**, as we found for the electrostatic field, since

$$\nabla \times \vec{H} = \vec{J}$$



Current density is a vector $\neq 0$

We define a **magnetic vector potential** \vec{A} through the relation

$$\nabla \times \vec{A} = \vec{B} = \mu \vec{H}$$

This definition automatically satisfies the condition of zero divergence for the induction field, since

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$



The divergence of a curl is always $= 0$

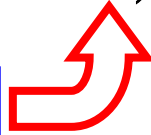
The **vector potential** can be introduced in the curl equation for the magnetic field

$$\nabla \times \vec{H} = \frac{1}{\mu} \nabla \times \vec{B} = \frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) = \frac{1}{\mu} \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \vec{J}$$

However, in order to **completely** specify the magnetic vector potential, we need to specify also its **divergence**. First, we observe that the definition of the vector potential is **not unique** since:

$$\nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla \psi) = \nabla \times \vec{A} + \nabla \times (\nabla \psi) = \nabla \times \vec{A}$$

ψ = Scalar function



Always = 0

In the **magnetostatic** case it is sufficient to specify (in physics terminology: to **choose the gauge**)

$$\nabla \cdot \vec{A} = 0$$

so that

$$\nabla \cdot \vec{A} = \nabla \cdot (\vec{A} + \nabla \psi) = \nabla \cdot \vec{A} + \nabla \cdot \nabla \psi = \nabla \cdot \vec{A} + \nabla^2 \psi = 0$$

We simply need to make sure that the arbitrary function ψ satisfies

$$\nabla^2 \psi = 0$$

We can then **simplify** the previous result for the curl equation to

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

the **magnetic equivalent** of the electrostatic Poisson equation.

The general solution of this **vector Laplacian equation** is given by

$$\vec{A}(x, y, z) = \frac{\mu}{4\pi} \int_V \frac{\vec{J}(x', y', z')}{r} dV$$

which is similar to the formal solution obtained before for the electrostatic potential for a distributed charge. If the current is confined to a wire with **cross-sectional** area S and described by a **curvilinear** coordinate l , we can write

$$|\vec{I}| = I = \vec{J} \cdot \vec{S} \quad dV = \vec{S} \cdot d\vec{l}$$

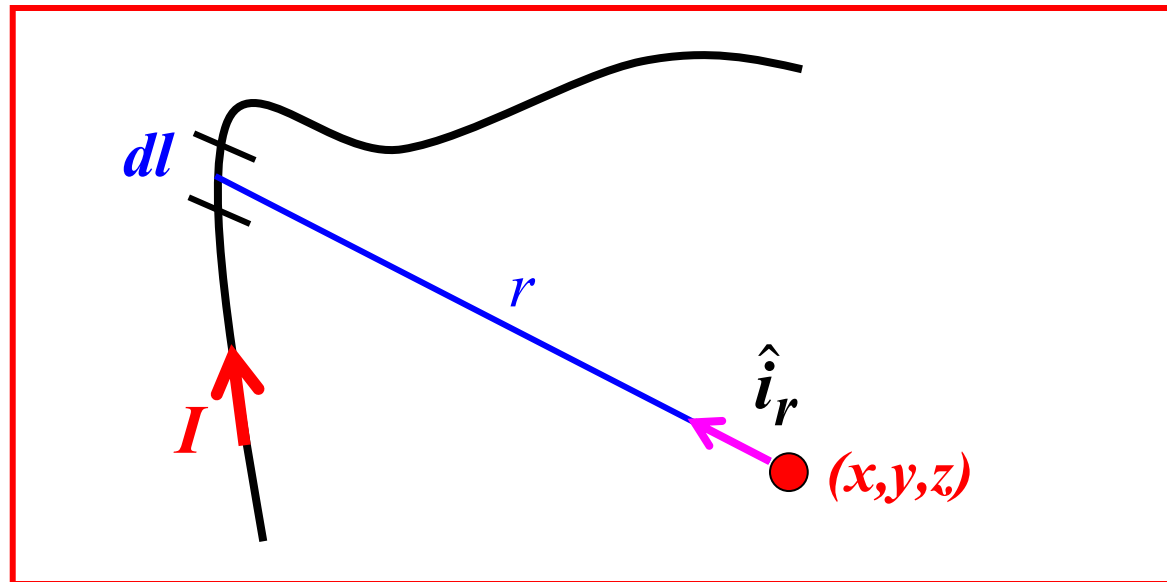
with a final result

$$\vec{A}(x, y, z) = \frac{\mu I}{4\pi} \int_l \frac{d\vec{l}}{r}$$

(note that the total current I is constant at any wire location).

The solution for the **magnetic field** obtained from the vector potential leads to the famous **Biot-Savart law**:

$$\begin{aligned}\vec{H} &= \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \times \vec{A} = \nabla \times \left(\frac{I}{4\pi} \int_l \frac{d\vec{l}}{r} \right) = \frac{I}{4\pi} \int_l \nabla \times \frac{d\vec{l}}{r} \\ &= -\frac{I}{4\pi} \int_l \left(\nabla \frac{1}{r} \right) \times d\vec{l} = \frac{I}{4\pi} \int_l \frac{d\vec{l} \times \hat{i}_r}{r^2}\end{aligned}$$



The **magnetic field** can also be determined by **direct integration** of the curl equation over a surface

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I$$

Stoke's theorem can be used to transform the left hand side of the equation, to obtain the integral form of **Ampere's law**

$$\int_l \vec{H} \cdot d\vec{l} = I$$

In many applications it is useful to determine the **magnetic flux** through a given surface. The **vector potential** can be used to modify a surface integral into a surface integral, using again **Stoke's theorem**

$$\text{Magnetic Flux} = \int_S \vec{B} \cdot d\vec{S} = \int_S \nabla \times \vec{A} \cdot d\vec{S} = \int_l \vec{A} \cdot d\vec{l}$$