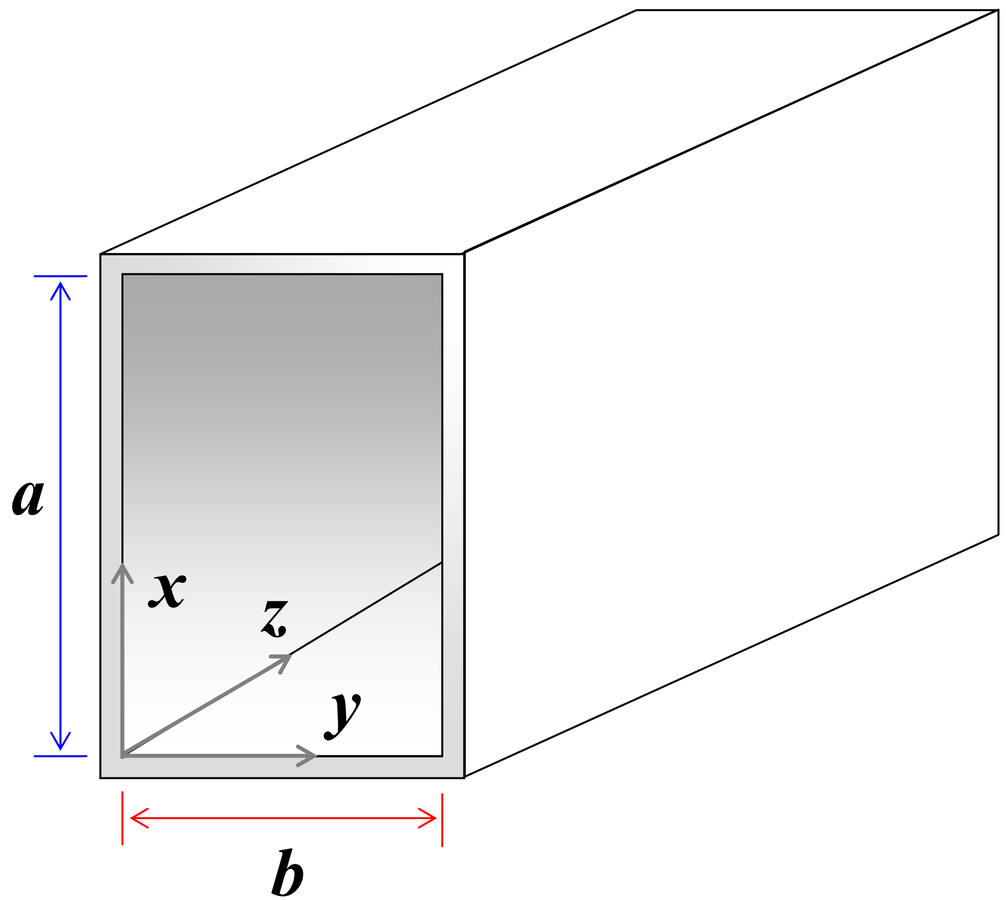
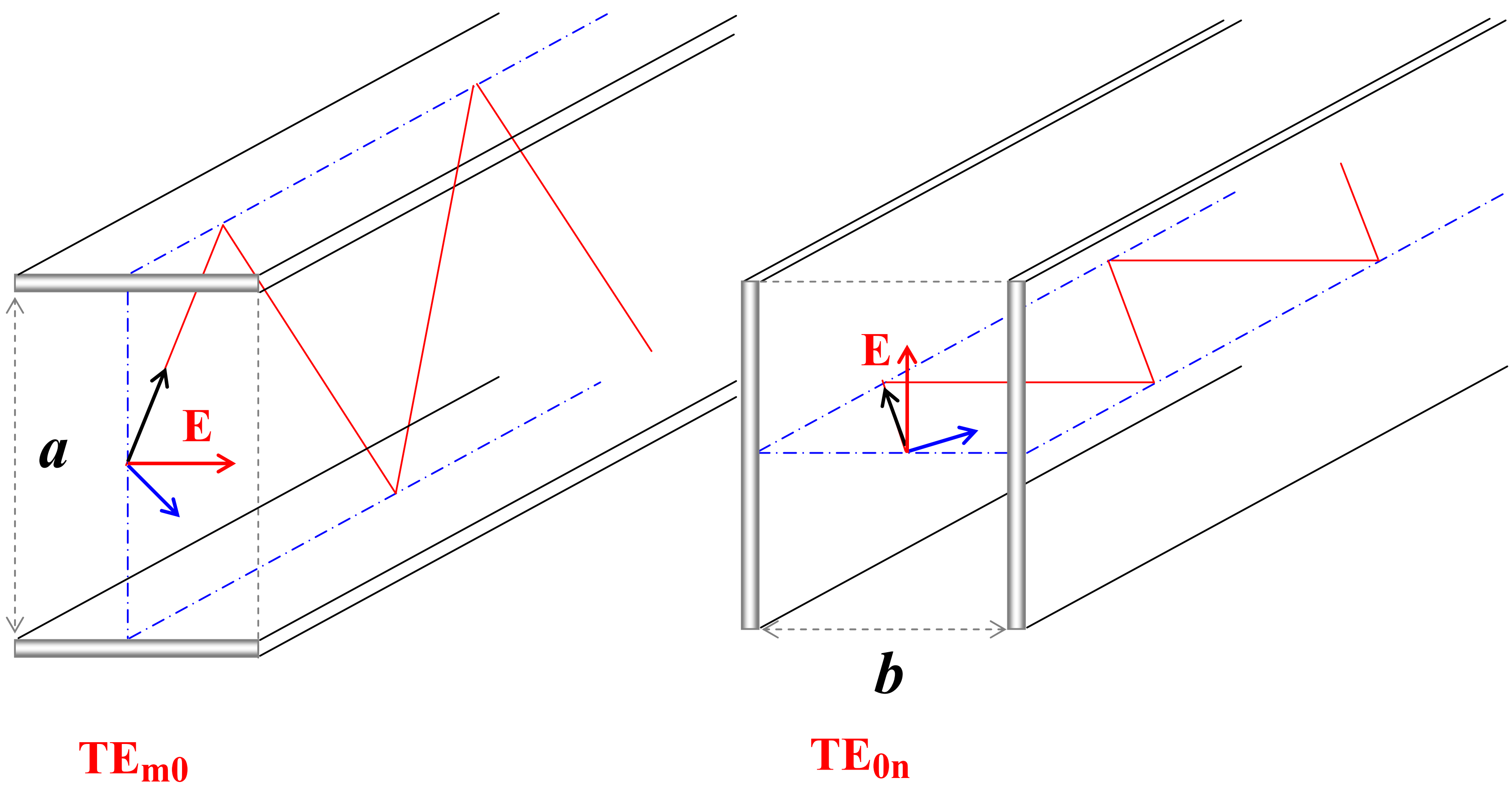
Rectangular Wave Guide



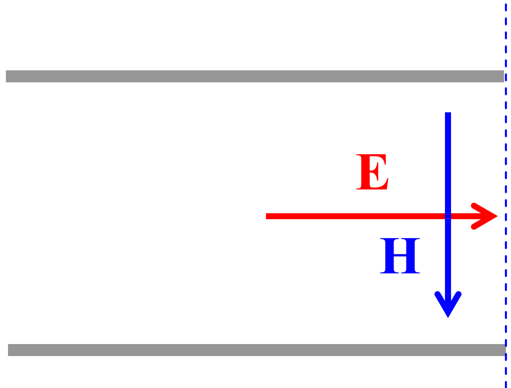
Assume perfectly conducting walls and perfect dielectric filling the wave guide.

Convention : a is always the wider side of the wave guide.

It is useful to consider the parallel plate wave guide as a starting point. The rectangular wave guide has the same TE modes corresponding to the two parallel plate wave guides obtained by considering opposite metal walls

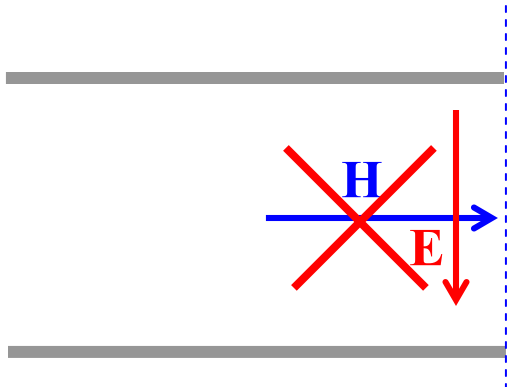


The TE modes of a parallel plate wave guide are preserved if perfectly conducting walls are added perpendicularly to the electric field.

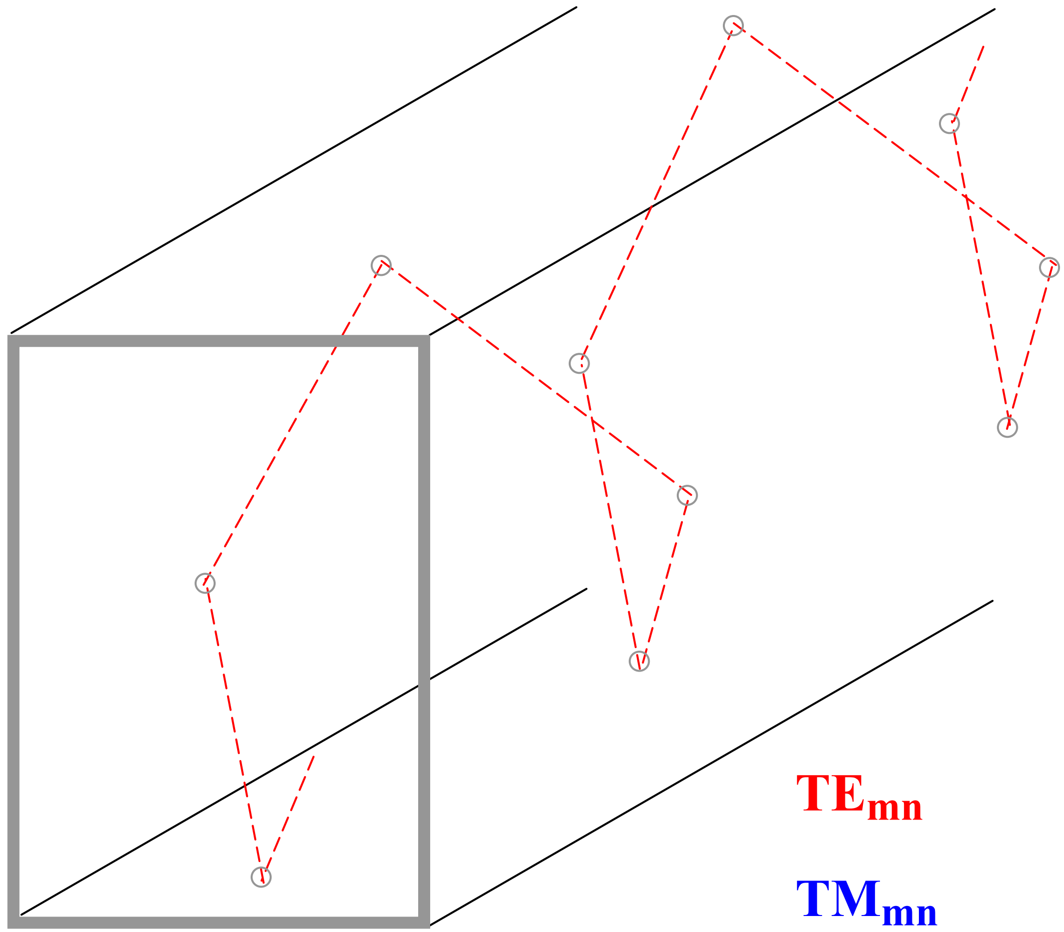


The added metal plate does not disturb normal electric field and tangent magnetic field.

On the other hand, TM modes of a parallel plate wave guide disappear if perfectly conducting walls are added perpendicularly to the magnetic field.



The magnetic field cannot be normal and the electric field cannot be tangent to a perfectly conducting plate.



The remaining modes are TE and TM modes bouncing off each wall, all with non-zero indices.

We have the following propagation vector components for the modes in a rectangular waveguide

$$\beta^2 = \omega^2 \mu \varepsilon = \beta_x^2 + \beta_y^2 + \beta_z^2$$

$$\beta_x = \frac{m\pi}{a} \quad ; \qquad \beta_y = \frac{n\pi}{b}$$

$$\beta_z^2 = \left(\frac{2\pi}{\lambda_z}\right)^2 = \left(\frac{2\pi}{\lambda_g}\right)^2 = \omega^2 \mu \varepsilon - \beta_x^2 - \beta_y^2$$

$$= \omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

At cut-off we have

$$\beta_z^2 = 0 = \left(2\pi f_c\right)^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

The cut-off frequencies for all modes are

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

with cut-off wavelengths

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

with indices

***TE* modes**	$m = 0, 1, 2, 3, \ldots$ $n = 0, 1, 2, 3, \ldots$	***TM* modes**	$m = 1, 2, 3, \ldots$ $n = 1, 2, 3, \ldots$

(but $m = n = 0$ not allowed)

The guide wavelengths and guide phase velocities are

$$\lambda_g = \lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{\sqrt{\omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} =$$

$$= \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{1}{\sqrt{\mu \varepsilon}} \frac{1}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}} = \frac{1}{\sqrt{\mu \varepsilon}} \frac{1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

The fundamental mode is the TE$_{10}$ with cut-off frequency

$$f_c\left(TE_{10}\right) = \frac{m}{2a\sqrt{\mu\,\varepsilon}}$$

The TE$_{10}$ electric field has only the y-component. From Ampere's law

$$\nabla \times \vec{\mathbf{E}} = -j\omega\,\mu\,\vec{\mathbf{H}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{E}_x = 0 & \mathbf{E}_y & \mathbf{E}_z = 0 \end{bmatrix} \Rightarrow \begin{array}{l} \cancel{\dfrac{\partial}{\partial y}\mathbf{E}_z} - \dfrac{\partial}{\partial z}\mathbf{E}_y = -j\omega\,\mu\,\mathbf{H}_x \\[2ex] \cancel{\dfrac{\partial}{\partial z}\mathbf{E}_x} - \cancel{\dfrac{\partial}{\partial x}\mathbf{E}_z} = -j\omega\,\mu\,\mathbf{H}_y = 0 \\[2ex] \dfrac{\partial}{\partial x}\mathbf{E}_y - \cancel{\dfrac{\partial}{\partial y}\mathbf{E}_x} = -j\omega\,\mu\,\mathbf{H}_z \end{array}$$

The complete field components for the TE$_{10}$ mode are then

$$\mathrm{E}_y = E_o \sin\left(\frac{\pi x}{a}\right) \; e^{-j\beta_z \cdot z}$$

$$\mathrm{H}_x = \frac{1}{j\omega\mu}\frac{\partial \mathrm{E}_y}{\partial z} = \frac{-j\beta_z}{j\omega\mu}\mathrm{E}_y = -\frac{\beta_z}{\omega\mu} E_o \sin\left(\frac{\pi x}{a}\right) \; e^{-j\beta_z \cdot z}$$

$$\mathrm{H}_z = -\frac{1}{j\omega\mu}\frac{\partial \mathrm{E}_x}{\partial z} = \frac{j}{\omega\mu}\frac{\pi}{a} E_o \cos\left(\frac{\pi x}{a}\right) \; e^{-j\beta_z \cdot z}$$

with

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2}$$

The time-average power density is given by the Poynting vector

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{2}\,\mathrm{Re}\left\{ \vec{\mathrm{E}} \times \vec{\mathrm{H}}^{*} \right\} = \frac{1}{2}\,\mathrm{Re}\{ \underbrace{E_o \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_z \cdot z}\ \vec{i}_y}_{\vec{\mathrm{E}}} \times$$

$$\underbrace{\left(-\frac{\beta_z}{\omega\,\mu} E_o^{*} \sin\left(\frac{\pi x}{a}\right) e^{j\beta_z \cdot z}\ \vec{i}_x - \frac{j}{\omega\,\mu}\frac{\pi}{a} E_o^{*} \cos\left(\frac{\pi x}{a}\right) e^{j\beta_z \cdot z}\ \vec{i}_z\right)}_{\vec{\mathrm{H}}^{*}} \}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{ \frac{\left|E_o\right|^2 \beta_z}{\omega\,\mu} \sin^2\left(\frac{\pi x}{a}\right)\ \vec{i}_z - \cancel{j\frac{\left|E_o\right|^2 \pi}{\omega\,\mu\,a} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) \vec{i}_x} \right\}$$

$$= \frac{\left|E_o\right|^2 \beta_z}{2\omega\,\mu} \sin^2\left(\frac{\pi x}{a}\right)\ \vec{i}_z$$

The resulting time-average power density flow is space-dependent on the cross-section (varying along x, uniform along y)

$$\left\langle \vec{P}(t) \right\rangle = \frac{|E_o|^2 \beta_z}{2\omega\mu} \sin^2\left(\frac{\pi x}{a}\right) \vec{i}_z$$

The total transmitted power for the TE_{10} mode is obtained by integrating over the cross-section of the rectangular wave guide

$$\left\langle P_{tot}(t) \right\rangle = \int_0^a dx \underbrace{\int_0^b dy}_{=b} \frac{|E_o|^2 \beta_z}{2\omega\mu} \sin^2\left(\frac{\pi x}{a}\right) = \frac{|E_o|^2 \beta_z}{2\omega\mu} b \frac{a}{\pi} \int_0^\pi \sin^2(u)\, du =$$

$$= \frac{|E_o|^2 \beta_z}{2\omega\mu} b \frac{ab}{\pi} \left[\frac{1}{2}u - \cancel{\frac{1}{4}\sin 2u}\right]_0^\pi = \frac{|E_o|^2 \beta_z}{4\omega\mu} ab = \frac{1}{2} \underbrace{\frac{|E_o|^2}{2}}_{\substack{\text{average} \\ |\mathrm{E}(x,y)|^2}} \underbrace{\frac{\beta_z}{\omega\mu}}_{1/\eta_{TE}} \underbrace{ab}_{\text{area}}$$

The rectangular waveguide has a high-pass behavior, since signals can propagate only if they have frequency higher than the cut-off for the TE_{10} mode.

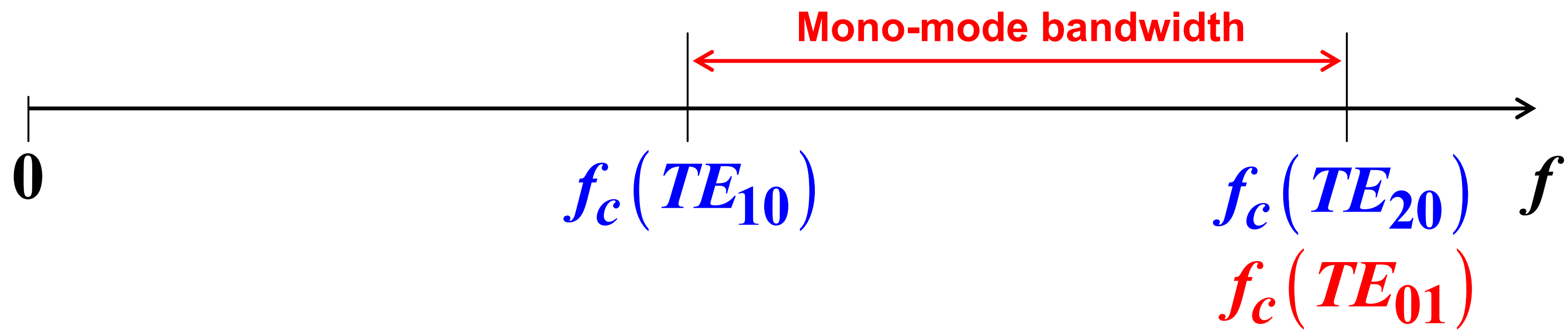
For mono-mode (or single-mode) operation, only the fundamental TE_{10} mode should be propagating over the frequency band of interest.

The mono-mode bandwith depends on the cut-off frequency of the second propagating mode. We have two possible modes to consider, TE_{01} and TE_{20}

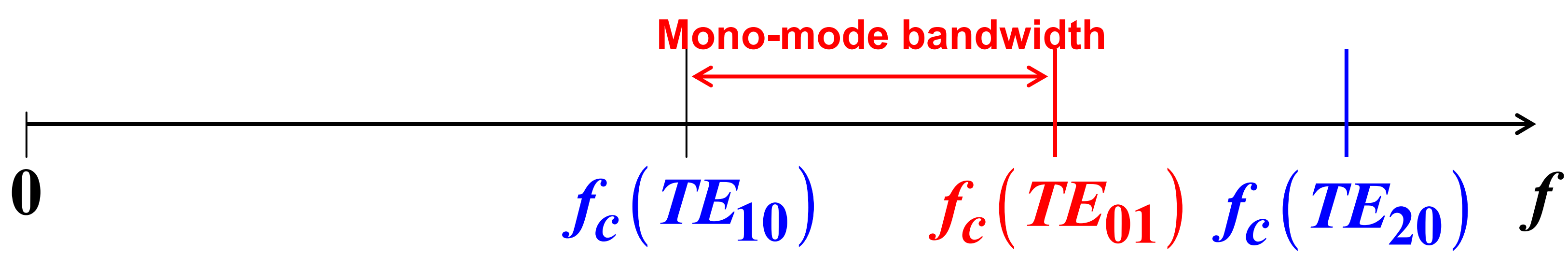
$$f_c\left(TE_{01}\right)=\frac{1}{2b\sqrt{\mu\varepsilon}}$$

$$f_c\left(TE_{20}\right)=\frac{1}{a\sqrt{\mu\varepsilon}}=2f_c\left(TE_{10}\right)$$

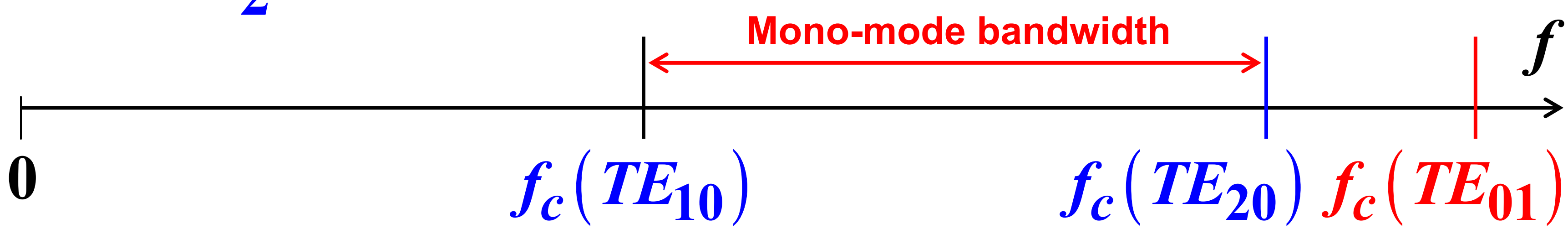
If $\quad b = \dfrac{a}{2} \quad \Rightarrow \quad f_c\left(TE_{01}\right) = f_c\left(TE_{20}\right) = 2 f_c\left(TE_{10}\right) = \dfrac{1}{a\sqrt{\mu\,\varepsilon}}$



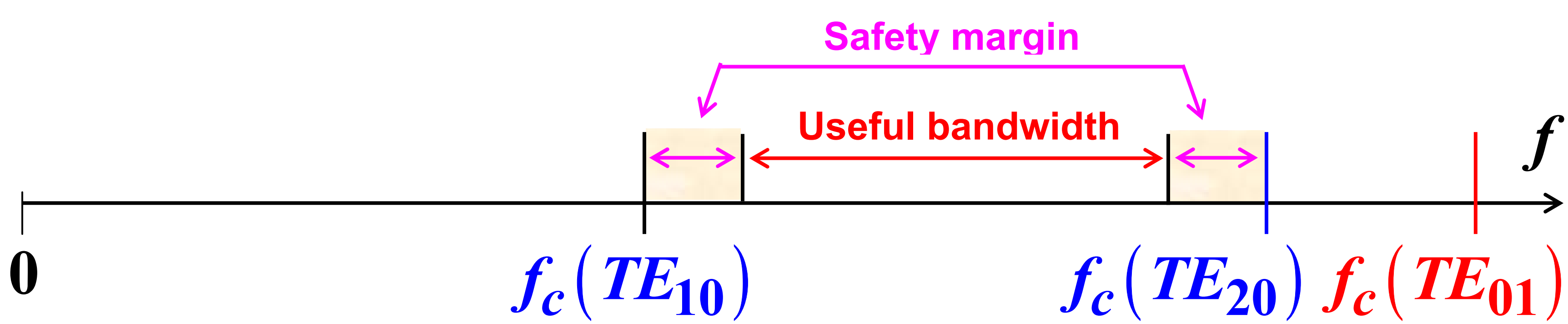
If $\quad a > b > \dfrac{a}{2} \quad \Rightarrow \quad f_c\left(TE_{10}\right) < f_c\left(TE_{01}\right) < f_c\left(TE_{20}\right)$



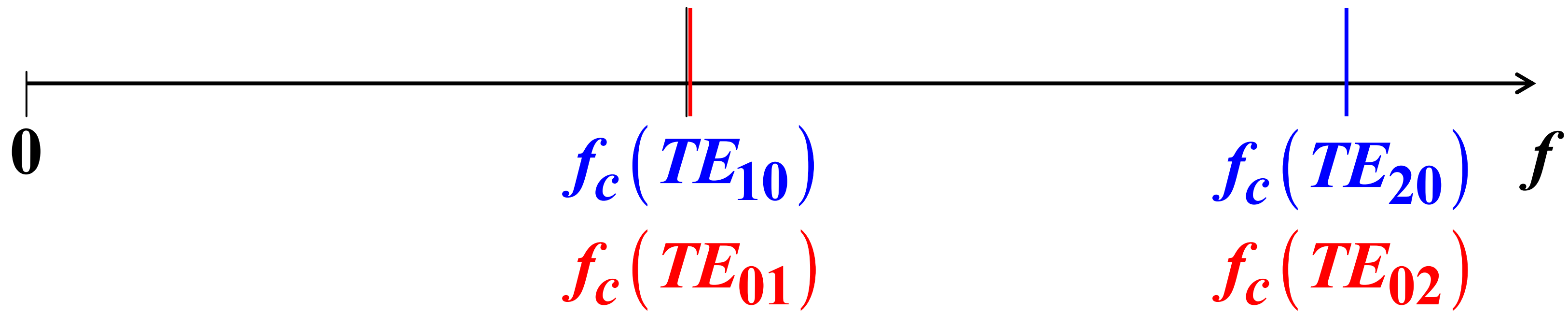
If $\quad b < \dfrac{a}{2} \quad \Rightarrow \quad f_c\left(TE_{20}\right) < f_c\left(TE_{01}\right)$



In practice, a safety margin of about 20% is considered, so that the useful bandwidth is less than the maximum mono-mode bandwidth. This is necessary to make sure that the first mode (TE_{10}) is well above cut-off, and the second mode (TE_{01} or TE_{20}) is strongly evanescent.



If $a = b$ (square wave guide) $\Rightarrow$ $f_c\left(TE_{10}\right) = f_c\left(TE_{01}\right)$



In the case of perfectly square wave guide, TE_{m0} and TE_{0n} modes with $m=n$ are degenerate with the same cut-off frequency.

Except for orthogonal field orientation, all other properties of degenerate modes are the same.

Example - Design an air-filled rectangular waveguide for the following operation conditions:

a) 10 GHz is the middle of the frequency band (single-mode operation)

b) b = a/2

The fundamental mode is the TE_{10} with cut-off frequency

$$f_c(TE_{10}) = \frac{1}{2a\sqrt{\varepsilon_o \mu_o}} = \frac{c}{2a} \approx \frac{3 \times 10^8}{2a} \text{ Hz}$$

For b=a/2, TE_{01} and TE_{20} have the same cut-off frequency.

$$f_c(TE_{01}) = \frac{1}{2b\sqrt{\varepsilon_o \mu_o}} = \frac{c}{2b} = \frac{c}{2}\frac{2}{a} = \frac{c}{a} \approx \frac{3 \times 10^8}{a} \text{ Hz}$$

$$f_c(TE_{20}) = \frac{1}{a\sqrt{\varepsilon_o \mu_o}} = \frac{c}{a} \approx \frac{3 \times 10^8}{a} \text{ Hz}$$

The operation frequency can be expressed in terms of the cut-off frequencies

$$f = f_c(TE_{10}) + \frac{f_c(TE_{01}) - f_c(TE_{10})}{2}$$

$$= \frac{f_c(TE_{10}) + f_c(TE_{01})}{2} = 10.0\ GHz$$

$$\Rightarrow 10.0 \times 10^{9} = \frac{1}{2}\left[\frac{3 \times 10^{8}}{2a} + \frac{3 \times 10^{8}}{a}\right]$$

$$\Rightarrow a = 2.25 \times 10^{-2}\, m \qquad b = \frac{a}{2} = 1.125 \times 10^{-2}\, m$$

Maxwell's equations for TE modes

Since the electric field must be transverse to the direction of propagation for a TE mode, we assume

$$E_z = 0$$

In addition, we assume that the wave has the following behavior along the direction of propagation

$$e^{-j\beta_z \cdot z}$$

In the general case of TE_{mn} modes it is more convenient to start from an assumed intensity of the z-component of the magnetic field

$$\begin{aligned} H_z &= H_o \cos(\beta_x \cdot x)\cos(\beta_y \cdot y)\, e^{-j\beta_z \cdot z} \\ &= H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_z \cdot z} \end{aligned}$$

Faraday's law for a TE mode, under the previous assumptions, is

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}$$

$$\Downarrow$$

$$\det \begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & 0 \end{bmatrix} \Rightarrow \begin{aligned} & -\frac{\partial}{\partial z} E_y = j\beta_z E_y = -j\omega\mu H_x \quad (1) \\ & \frac{\partial}{\partial z} E_x = -j\beta_z E_x = -j\omega\mu H_y \quad (2) \\ & \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x = -j\omega\mu H_z \quad (3) \end{aligned}$$

Ampere's law for a TE mode, under the previous assumptions, is

$$\nabla \times \vec{\mathrm{H}} = j\omega\varepsilon\,\vec{\mathrm{E}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathrm{H}_x & \mathrm{H}_y & \mathrm{H}_z \end{bmatrix} \Rightarrow \begin{aligned} &\frac{\partial}{\partial y}\mathrm{H}_z + j\beta_z\mathrm{H}_y = j\omega\varepsilon\mathrm{E}_x \quad (4) \\ &-j\beta_z\mathrm{H}_x - \frac{\partial}{\partial x}\mathrm{H}_z = j\omega\varepsilon\mathrm{E}_y \quad (5) \\ &\frac{\partial}{\partial x}\mathrm{H}_y - \frac{\partial}{\partial y}\mathrm{H}_x = j\omega\varepsilon\mathrm{E}_z = 0 \quad (6) \end{aligned}$$

From (1) and (2) we obtain the characteristic wave impedance for the TE modes

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta_z} = \eta_{TE}$$

At cut-off

$$\beta_z = 0 \quad \Rightarrow \quad 2 f_c \sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$f_c = \frac{v_p}{\lambda_c} = \frac{1}{\lambda_c\sqrt{\mu\varepsilon}} \quad \Rightarrow \quad \lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

In general,

$$\beta_z = \sqrt{\omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \frac{2\pi}{\lambda}\sqrt{1 - \frac{\lambda^2}{(2\pi)^2}\frac{4}{\lambda_c^2}}$$

$$\Rightarrow \quad \beta_z = \frac{2\pi}{\lambda}\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}$$

and we obtain an alternative expression for the characteristic wave impedance of TE modes as

$$\boxed{\eta_{TE} = \frac{\omega\mu}{\beta_z} = \eta_o \left(1 - \left(\frac{\lambda}{\lambda_c}\right)^2\right)^{-1/2}}$$

From (4) and (5) we obtain

$$\frac{\partial}{\partial y}\mathrm{H}_z + j\beta_z \mathrm{H}_y = j\omega\varepsilon \mathrm{E}_x = j\omega\varepsilon \cdot \eta_{TE}\mathrm{H}_y$$

$$\mathrm{H}_y = \frac{1}{j\omega\varepsilon\cdot\eta_{TE} - j\beta_z}\frac{\partial \mathrm{H}_z}{\partial y} = \frac{1}{j\omega\varepsilon\cdot\dfrac{\omega\mu}{\beta_z} - j\beta_z}\frac{\partial \mathrm{H}_z}{\partial y}$$

$$\Rightarrow \quad \mathrm{H}_y = -\frac{j\beta_z}{\beta^2 - \beta_z^2}\frac{\partial \mathrm{H}_z}{\partial y} = -j\beta_z\left(\frac{\lambda_c}{2\pi}\right)^2\frac{\partial \mathrm{H}_z}{\partial y}$$

$$-j\beta_z \mathrm{H}_x - \frac{\partial}{\partial x}\mathrm{H}_z = j\omega\varepsilon \mathrm{E}_y = -j\omega\varepsilon\,\eta_{TE}\mathrm{H}_x$$

$$\Rightarrow \quad \mathrm{H}_x = -\frac{j\beta_z}{\beta^2 - \beta_z^2}\frac{\partial \mathrm{H}_z}{\partial x} = -j\beta_z\left(\frac{\lambda_c}{2\pi}\right)^2\frac{\partial \mathrm{H}_z}{\partial x}$$

We have used

$$\frac{1}{\beta^2 - \beta_z^2} = \frac{1}{\beta_x^2 + \beta_y^2} = \frac{1}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \left(\frac{\lambda_c}{2\pi}\right)^2$$

The final expressions for the magnetic field components of TE modes in rectangular waveguide are

$$\boxed{\begin{aligned} \mathrm{H}_x &= j\beta_z \frac{m\pi}{a}\left(\frac{\lambda_c}{2\pi}\right)^2 H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z} \\ \mathrm{H}_y &= j\beta_z \frac{n\pi}{b}\left(\frac{\lambda_c}{2\pi}\right)^2 H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z} \\ \mathrm{H}_z &= H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z} \end{aligned}}$$

The final electric field components for TE modes in rectangular wave guide are

$$\boxed{\begin{aligned}
\mathrm{E}_x &= \eta_{TE}\mathrm{H}_y \\
&= j\eta_{TE}\beta_z \frac{n\pi}{b}\left(\frac{\lambda_c}{2\pi}\right)^2 H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z} \\
\mathrm{E}_y &= -\eta_{TE}\mathrm{H}_x \\
&= -j\eta_{TE}\beta_z \frac{m\pi}{a}\left(\frac{\lambda_c}{2\pi}\right)^2 H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z} \\
\mathrm{E}_z &= 0
\end{aligned}}$$

Maxwell's equations for TM modes

Since the magnetic field must be transverse to the direction of propagation for a TM mode, we assume

$$\mathrm{H}_z = 0$$

In addition, we assume that the wave has the following behavior along the direction of propagation

$$e^{-j\beta_z \cdot z}$$

In the general case of TM_{mn} modes it is more convenient to start from an assumed intensity of the z-component of the electric field

$$\mathrm{E}_z = E_o \cos\left(\beta_x \cdot x\right)\cos\left(\beta_y \cdot y\right)e^{-j\beta_z \cdot z}$$

$$= E_o \cos\left(\frac{m\pi}{a} x\right)\cos\left(\frac{n\pi}{b} y\right)e^{-j\beta_z \cdot z}$$

Faraday's law for a **TM** mode, under the previous assumptions, is

$$\nabla \times \vec{\mathrm{E}} = -j\omega\mu\vec{\mathrm{H}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathrm{E}_x & \mathrm{E}_y & \mathrm{E}_z \end{bmatrix} \Rightarrow \begin{aligned} &\frac{\partial}{\partial y}\mathrm{E}_z + j\beta_z\mathrm{E}_y = -j\omega\mu\mathrm{H}_x \quad (1) \\ &-j\beta_z\mathrm{E}_x - \frac{\partial}{\partial x}\mathrm{E}_z = -j\omega\mu\mathrm{H}_y \quad (2) \\ &\frac{\partial}{\partial x}\mathrm{E}_y - \frac{\partial}{\partial y}\mathrm{E}_x = -j\omega\mu\mathrm{H}_z \quad (3) \end{aligned}$$

Ampere's law for a TM mode, under the previous assumptions, is

$$\nabla \times \vec{H} = j\omega\varepsilon\,\vec{E}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & 0 \end{bmatrix} \Rightarrow \begin{array}{lr} j\beta_z H_y = j\omega\varepsilon E_x & (4) \\ -j\beta_z H_x = j\omega\varepsilon E_y & (5) \\ \dfrac{\partial}{\partial x}H_y - \dfrac{\partial}{\partial y}H_x = j\omega\varepsilon E_z & (6) \end{array}$$

From (4) and (5) we obtain the characteristic wave impedance for the TM modes

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta_z}{\omega\varepsilon} = \eta_{TM}$$

We can finally express the characteristic wave impedance alternatively as

$$\boxed{\eta_{TM} = \frac{\beta_z}{\omega\varepsilon} = \eta_o\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$$

Note once again that the same cut-off conditions, found earlier for TE modes, also apply for TM modes.

From (1) and (2) we obtain

$$\frac{\partial}{\partial y}\mathrm{E}_z + j\beta_z \mathrm{E}_y = -j\omega\mu \mathrm{H}_x = j\omega\mu\cdot\frac{\mathrm{E}_y}{\eta_{TM}}$$

$$\mathrm{E}_y = \frac{1}{j\omega\mu/\eta_{TM} - j\beta_z}\frac{\partial \mathrm{E}_z}{\partial y} = \frac{1}{j\omega\mu\cdot\dfrac{\omega\varepsilon}{\beta_z} - j\beta_z}\frac{\partial \mathrm{E}_z}{\partial y}$$

$$\Rightarrow \quad \mathrm{E}_y = -\frac{j\beta_z}{\beta^2-\beta_z^2}\frac{\partial \mathrm{E}_z}{\partial y} = -j\beta_z\left(\frac{\lambda_c}{2\pi}\right)^2\frac{\partial \mathrm{E}_z}{\partial y}$$

$$-j\beta_z \mathrm{E}_x - \frac{\partial}{\partial x}\mathrm{E}_z = -j\omega\mu \mathrm{H}_y = -j\omega\mu\frac{\mathrm{E}_x}{\eta_{TM}}$$

$$\Rightarrow \quad \mathrm{E}_x = -\frac{j\beta_z}{\beta^2-\beta_z^2}\frac{\partial \mathrm{E}_z}{\partial x} = -j\beta_z\left(\frac{\lambda_c}{2\pi}\right)^2\frac{\partial \mathrm{E}_z}{\partial x}$$

The final expressions for the electric field components of TM modes in rectangular waveguide are

$$\mathrm{E}_x = -j\beta_z \frac{m\pi}{a}\left(\frac{\lambda_c}{2\pi}\right)^2 E_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z}$$

$$\mathrm{E}_y = -j\beta_z \frac{n\pi}{b}\left(\frac{\lambda_c}{2\pi}\right)^2 E_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z}$$

$$\mathrm{E}_z = E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z \cdot z}$$

The final magnetic field components for TM modes in rectangular wave guide are

$$\mathrm{H}_x = -\mathrm{E}_y / \eta_{TM}$$

$$= j \frac{\beta_z}{\eta_{TM}} \frac{n\pi}{b} \left(\frac{\lambda_c}{2\pi} \right)^2 E_o \sin\left(\frac{m\pi}{a} x \right) \cos\left(\frac{n\pi}{b} y \right) e^{-j\beta_z \cdot z}$$

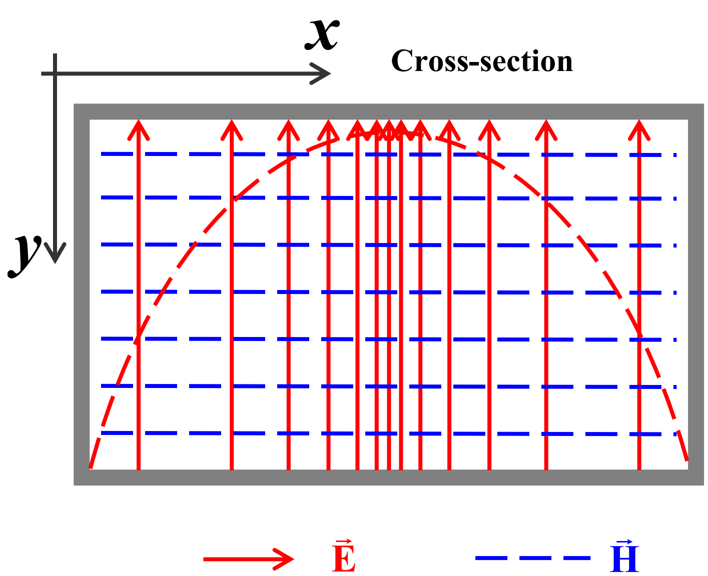
$$\mathrm{H}_y = \mathrm{E}_x / \eta_{TM}$$

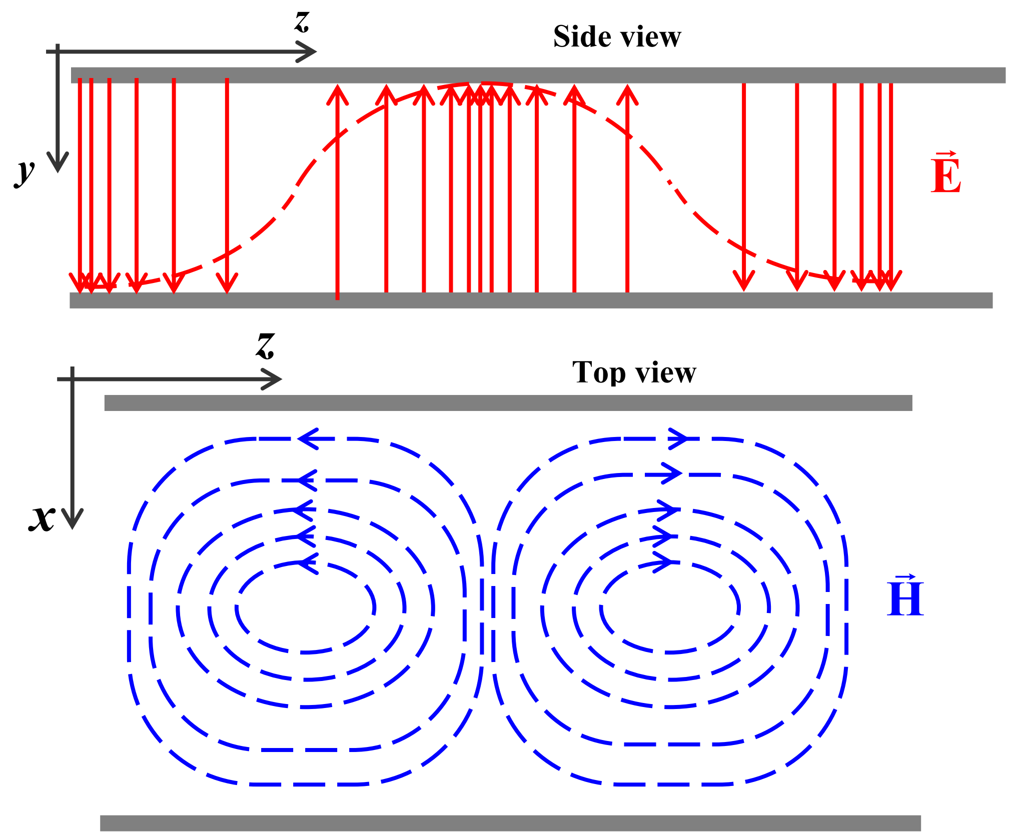
$$= -j \frac{\beta_z}{\eta_{TM}} \frac{m\pi}{a} \left(\frac{\lambda_c}{2\pi} \right)^2 E_o \cos\left(\frac{m\pi}{a} x \right) \sin\left(\frac{n\pi}{b} y \right) e^{-j\beta_z \cdot z}$$

$$\mathrm{H}_z = 0$$

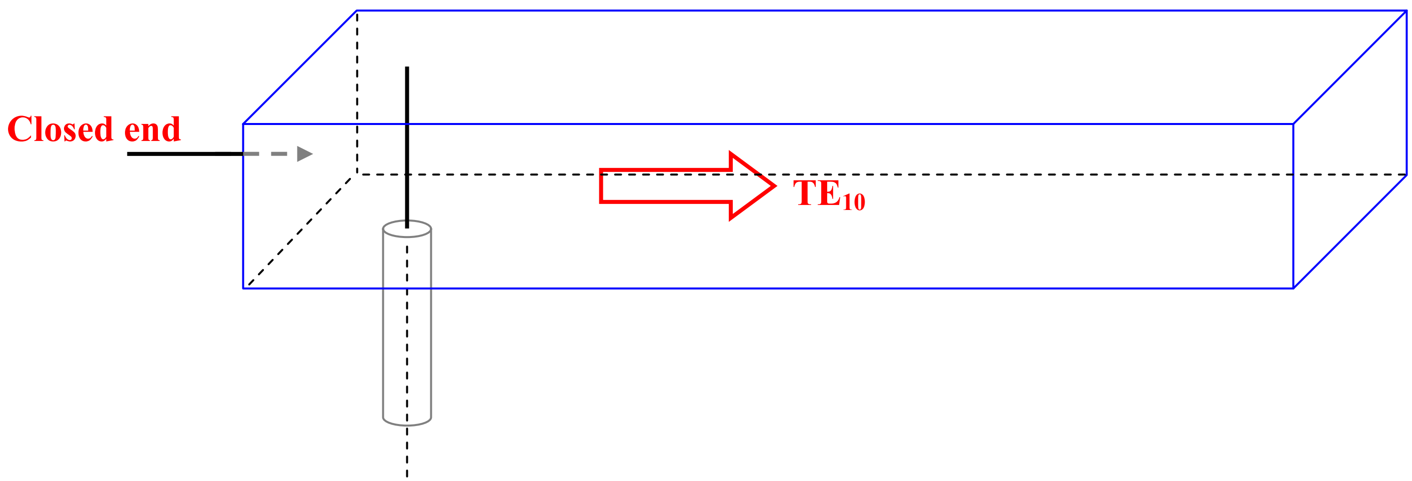
Note: all the TM field components are zero if either β_x=0 or β_y=0. This proves that TM_{mo} or TM_{on} modes cannot exist in the rectangular wave guide.

- **Field patterns** for the TE_{10} mode in rectangular wave guide





- **The simple arrangement below can be used to excite the TE_{10} in a rectangular waveguide.**



The inner conductor of the coaxial cable behaves like an antenna and it creates a maximum electric field in the middle of the cross-section.