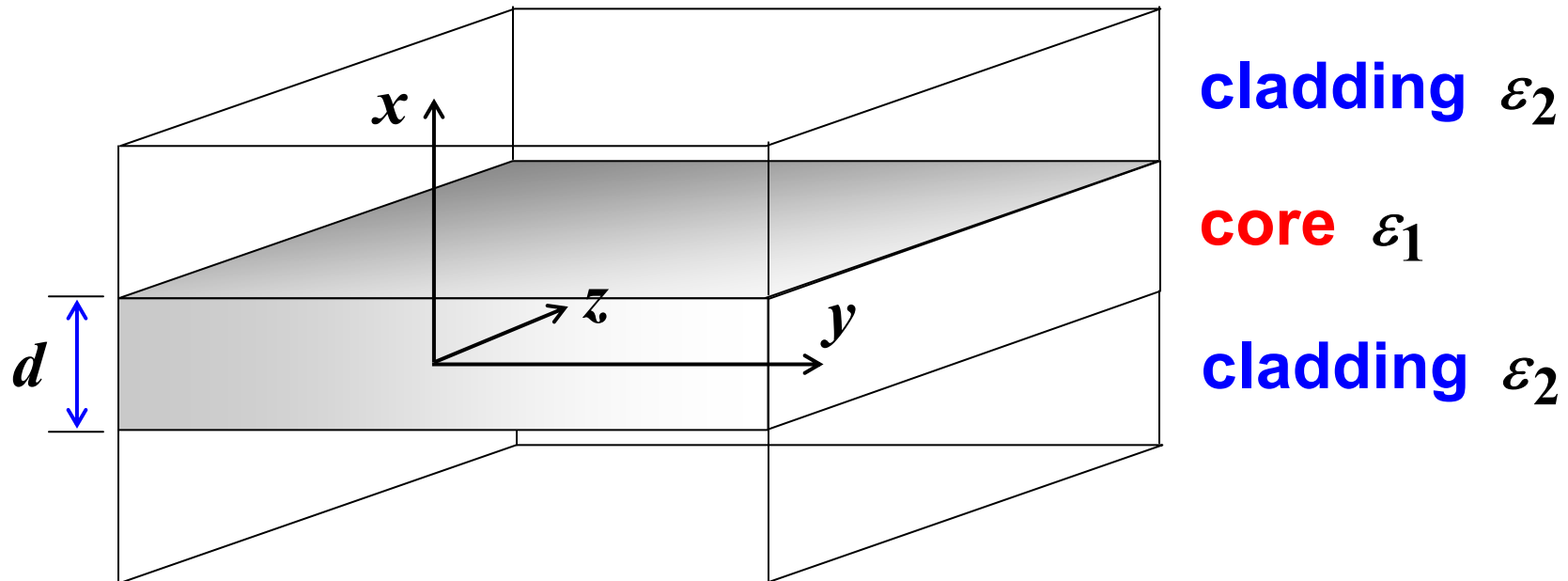


## Dielectric Wave Guide

A dielectric waveguide is a structure which exploits **total reflection** at dielectric interfaces to **guide** electromagnetic radiation.

The simplest case is the **symmetric dielectric slab wave guide**

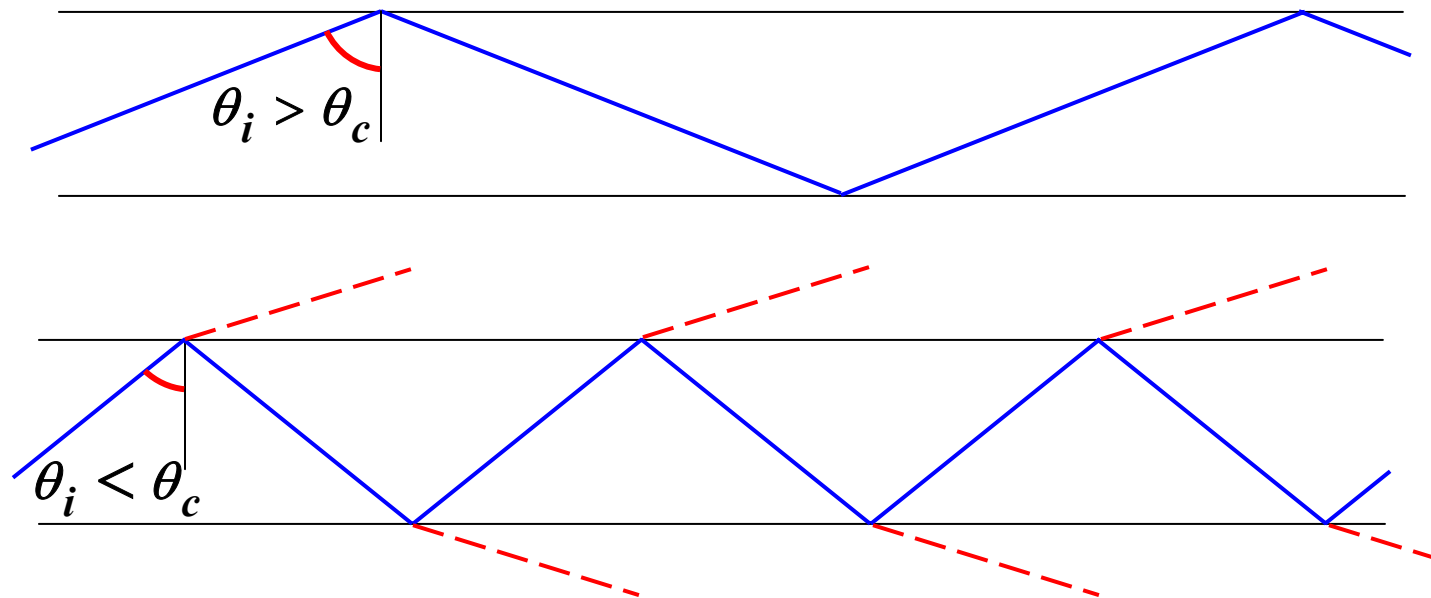


For **guidance** one must have

$$\epsilon_1 > \epsilon_2$$

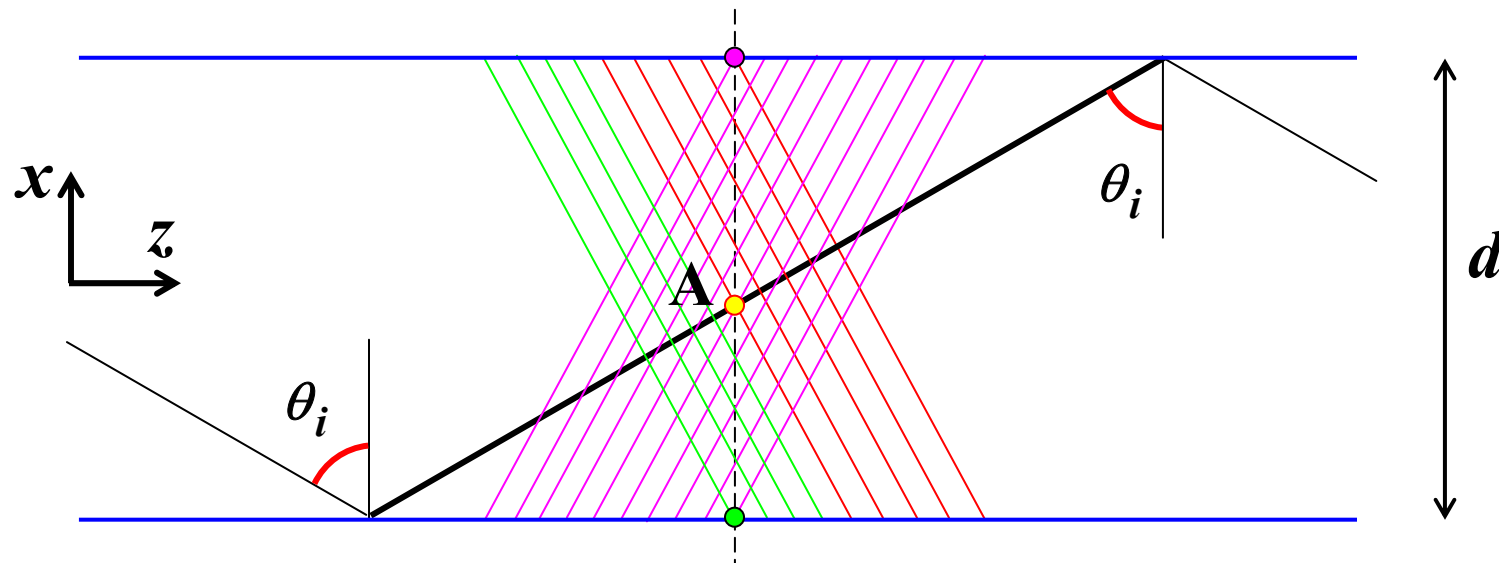
Similarly to the parallel plate waveguide, we assume **propagation** along the **z-direction** and **uniform** conditions along the **y-direction**.

**Guided waves** are launched with angle of incidence **larger** than the **critical angle**, so that **total (internal) reflection** takes place. Waves launched at smaller angles suffer partial refraction into the cladding and eventually the power in the core region will disappear for sufficiently long wave guides.



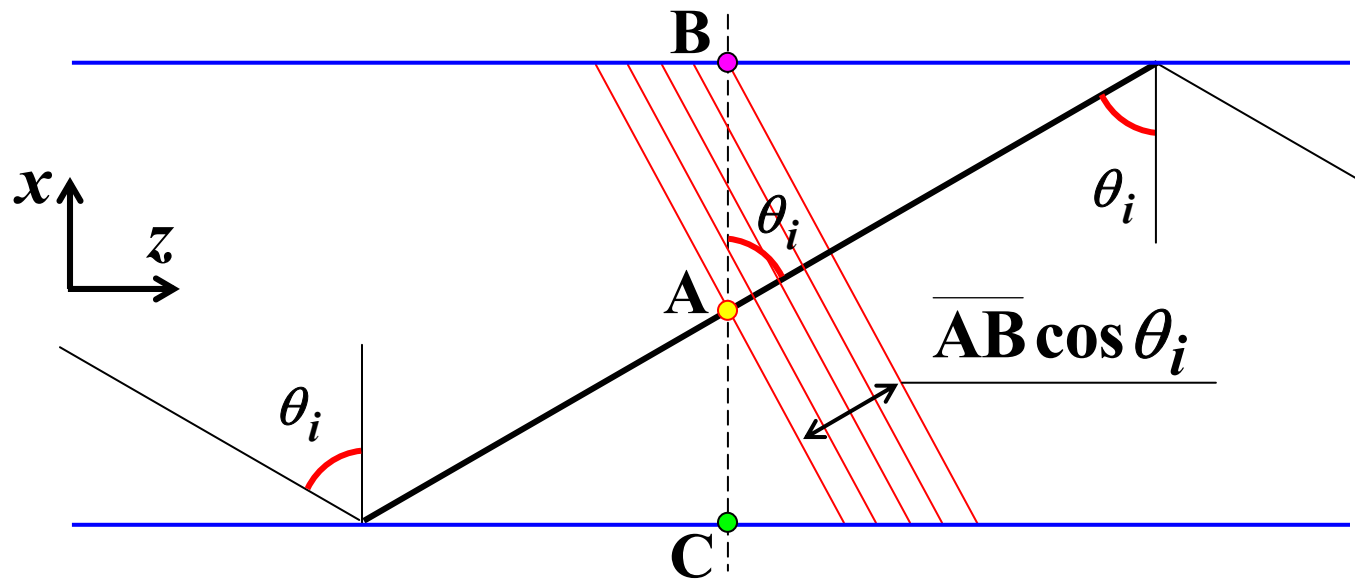
We consider again **TE** and **TM** modes. Only certain angles of incidence are allowed, but here the **reflection coefficient** for total reflection is a **complex** quantity, introducing a **phase shift** in the reflected field, which depends on the angle of incidence. In the case of metal plates, instead, there is always a phase shift of  $180^\circ$  for the tangential electric field.

In order for the **angle** to be **accepted**, the wave needs to establish a **self-consistent constructive interference** pattern for any point inside the core, as indicated in the figure below



Consider a point **A** in the core of the wave guide and a **wave front** moving from it reaching point **B**. The **phase shift** for the phase planes moving from **A** to **B** is

$$\Delta\varphi_1 = -\beta_1 \cdot \overline{AB} \cos \theta_i = -\frac{2\pi}{\lambda_0 / \sqrt{\epsilon_{r1}}} \overline{AB} \cos \theta_i$$



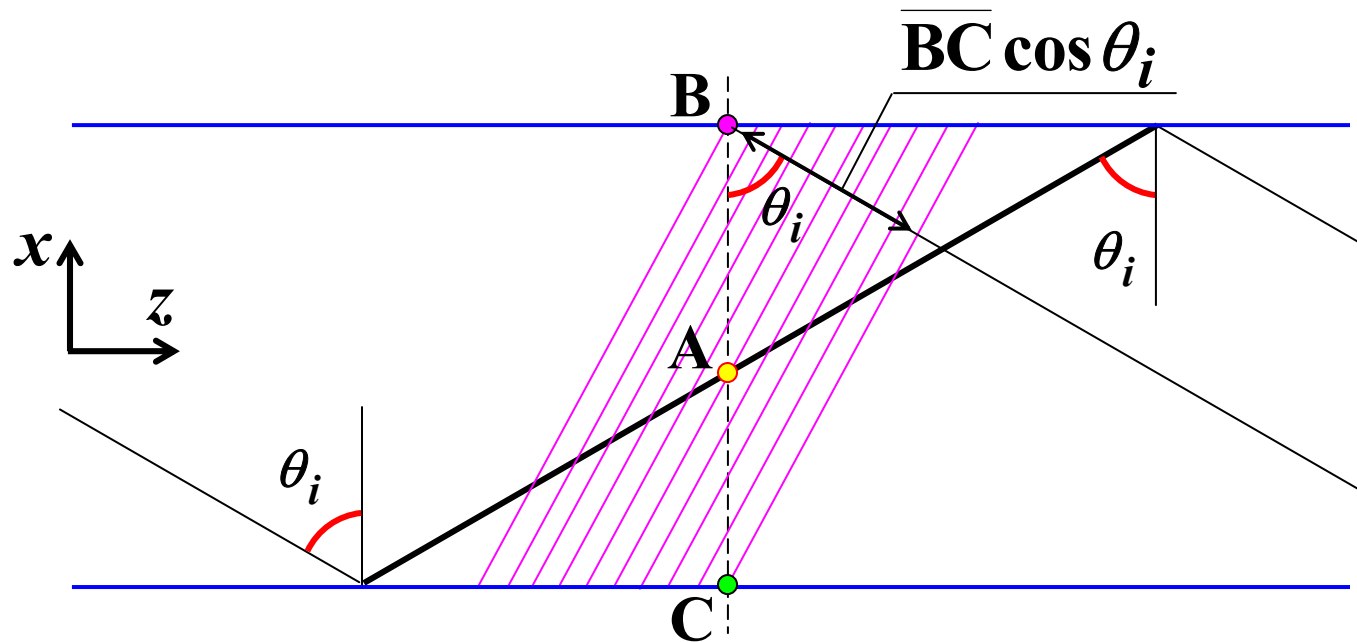
$\lambda_0$  is the **wavelength** in **vacuum** at the given frequency of operation.

The wave front reflected at point **B** experiences a phase jump equal to the phase of the **complex reflection coefficient**. Assuming a **TE** wave, or **perpendicular** polarization,

$$\begin{aligned}
 \Delta\varphi_2 = \angle \Gamma_{\perp}(E)_B &= \angle \left( \frac{\sqrt{\varepsilon_1} \cos \theta_i + j\sqrt{\varepsilon_2} \sqrt{\varepsilon_1 / \varepsilon_2 \sin^2 \theta_i - 1}}{\sqrt{\varepsilon_1} \cos \theta_i - j\sqrt{\varepsilon_2} \sqrt{\varepsilon_1 / \varepsilon_2 \sin^2 \theta_i - 1}} \right) \\
 &= \angle \left( \frac{\cos \theta_i + j\sqrt{\sin^2 \theta_i - \varepsilon_1 / \varepsilon_2}}{\cos \theta_i - j\sqrt{\sin^2 \theta_i - \varepsilon_1 / \varepsilon_2}} \right) \\
 &= 2 \angle \left( \cos \theta_i + j\sqrt{\sin^2 \theta_i - \varepsilon_1 / \varepsilon_2} \right) \\
 &= 2 \tan^{-1} \frac{\sqrt{\sin^2 \theta_i - \varepsilon_1 / \varepsilon_2}}{\cos \theta_i}
 \end{aligned}$$

Then, the reflected wave experiences a **phase shift** when moving from **B** to **C**

$$\Delta\varphi_3 = -\beta_1 \cdot \overline{BC} \cos \theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\epsilon_{r1}}} \overline{BC} \cos \theta_i$$



The wave front reflected at point **C** experiences again a phase jump equal to the phase of the **complex reflection coefficient**. For a symmetric waveguide

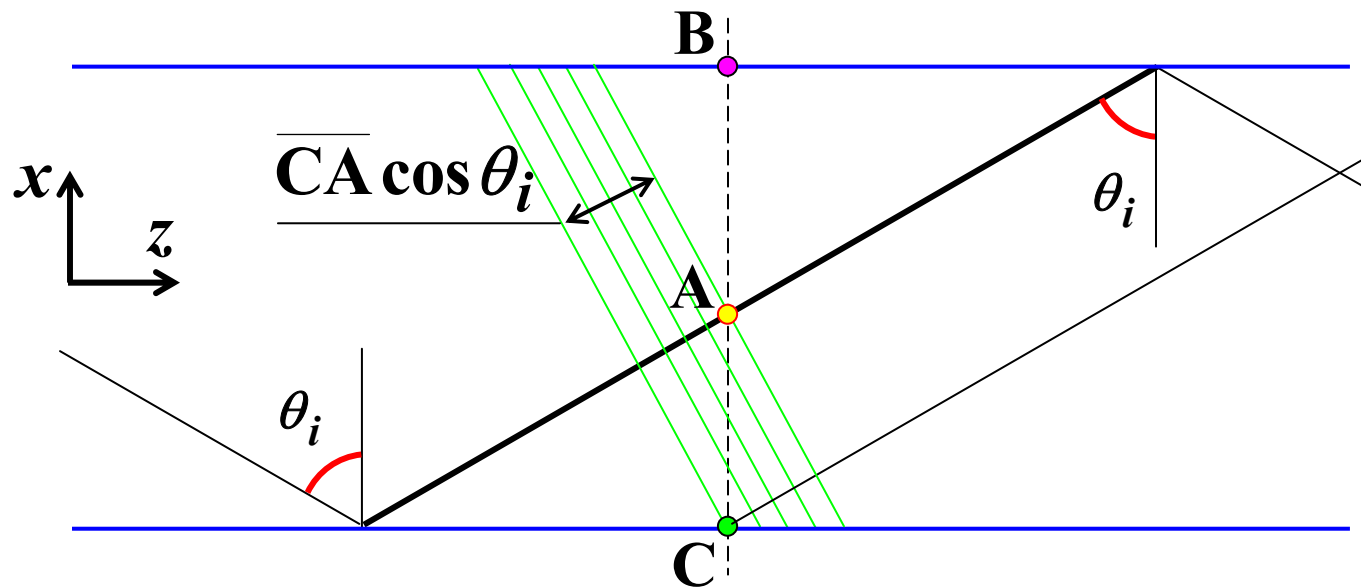
$$\Rightarrow \Delta\varphi_4 = \Delta\varphi_2$$

$$\Delta\varphi_4 = \angle \Gamma_{\perp}(E)_C = \angle \left( \frac{\sqrt{\varepsilon_1} \cos \theta_i + j\sqrt{\varepsilon_2} \sqrt{\frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_i - 1}}{\sqrt{\varepsilon_1} \cos \theta_i - j\sqrt{\varepsilon_2} \sqrt{\frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_i - 1}} \right)$$

$$= 2 \tan^{-1} \frac{\sqrt{\sin^2 \theta_i - \frac{\varepsilon_2}{\varepsilon_1}}}{\cos \theta_i}$$

The reflected wave experiences a **phase shift** moving from **C** back to **A**

$$\Delta\varphi_5 = -\beta_1 \cdot \overline{CA} \cos \theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\epsilon_{r1}}} \overline{CA} \cos \theta_i$$



For **constructive interference** (**self-consistency**), the sum of all the phase shift components must be equal to a **multiple of  $2\pi$**

$$-\frac{2\pi}{\lambda_o / \sqrt{\epsilon_{r1}}} (\overline{AB} + \overline{BC} + \overline{CA}) \cos \theta_i + \Delta\varphi_2 + \Delta\varphi_4 = -2m\pi,$$

$$m = 0, 1, 2 \dots$$

with  $(\overline{AB} + \overline{BC} + \overline{CA}) = 2d$

$$\Rightarrow \frac{2\pi d}{\lambda_o / \sqrt{\epsilon_{r1}}} \cos \theta_i - m\pi = 2 \tan^{-1} \frac{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}}{\cos \theta_i}$$

$$m = 0, 1, 2 \dots$$

Taking the tangent of all terms we obtain the **characteristic equation** for the **TE modes**.

$$\tan\left(\frac{\pi d \sqrt{\epsilon_{r1}}}{\lambda_0} \cos \theta_i - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}}{\cos \theta_i}, \quad m = 0, 1, 2, \dots$$

In terms of **even** and **odd** solutions, we can rewrite

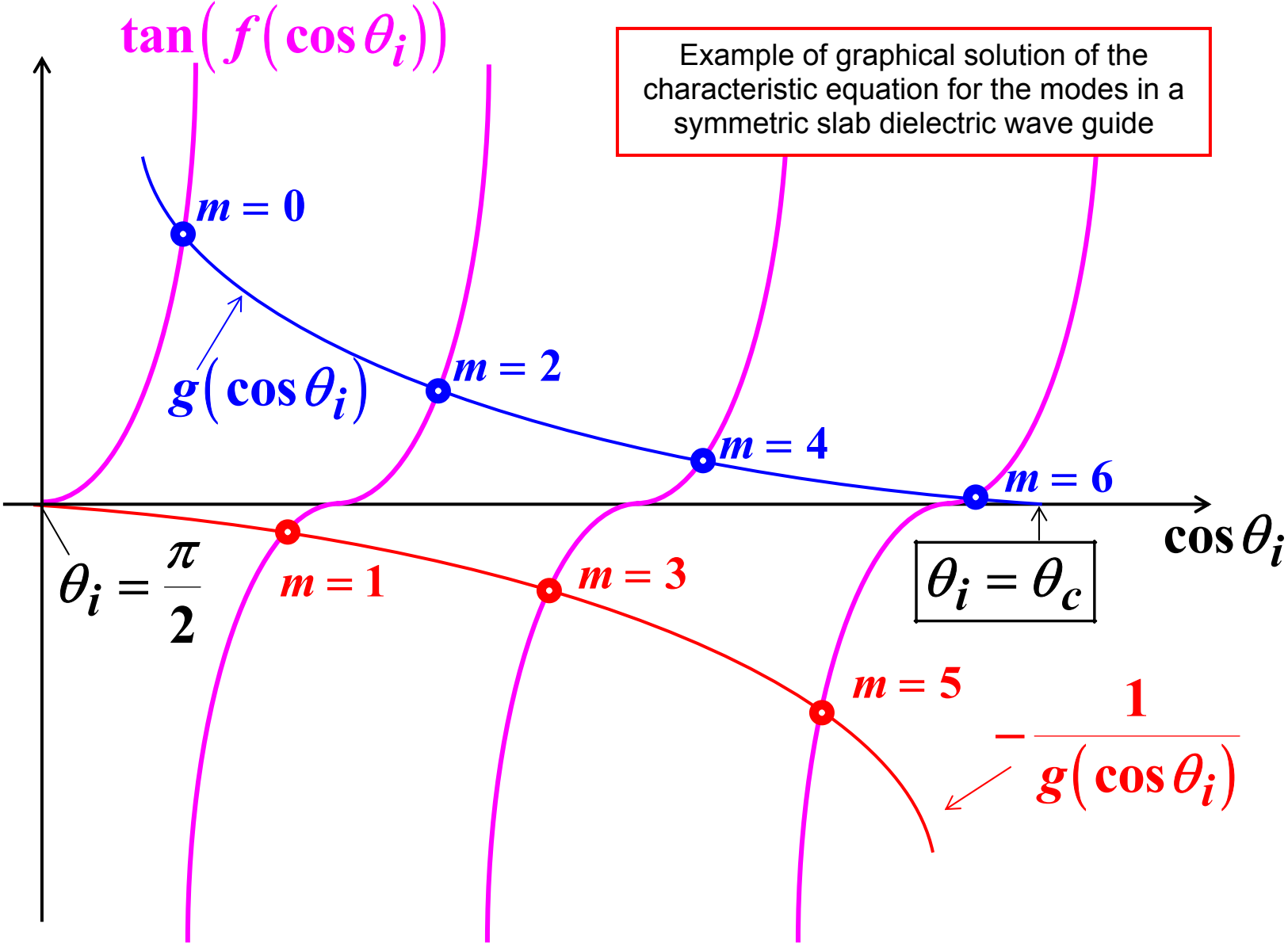
$$\tan\left(\underbrace{\frac{\pi d \sqrt{\epsilon_{r1}}}{\lambda_0} \cos \theta_i}_{f(\cos \theta_i)}\right) = \begin{cases} \frac{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}}{\cos \theta_i} = g(\cos \theta_i) & \text{Even modes} \\ & m = 0, 2, \dots \\ -\frac{\cos \theta_i}{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}} = -\frac{1}{g(\cos \theta_i)} & \text{Odd modes} \\ & m = 1, 3, \dots \end{cases}$$

The characteristic equation for **TM modes** is obtained by using the reflection coefficient for **parallel polarization** in the derivation

$$\tan\left(\frac{\pi d \sqrt{\epsilon_{r1}}}{\lambda_0} \cos \theta_i - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}}{(\epsilon_2 / \epsilon_1) \cos \theta_i}, \quad m = 0, 1, 2, \dots$$

or, in terms of **even** and **odd** solutions

$$\tan\left(\underbrace{\frac{\pi d \sqrt{\epsilon_{r1}}}{\lambda_0} \cos \theta_i}_{f(\cos \theta_i)}\right) = \begin{cases} \frac{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}}{(\epsilon_2 / \epsilon_1) \cos \theta_i} = g(\cos \theta_i) & \boxed{\text{Even modes}} \\ & m = 0, 2, \dots \\ -\frac{(\epsilon_2 / \epsilon_1) \cos \theta_i}{\sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}} = -\frac{1}{g(\cos \theta_i)} & \boxed{\text{Odd modes}} \\ & m = 1, 3, \dots \end{cases}$$



The **cut-off frequencies** for the modes are obtained by observing that at cut-off the angle of incidence is minimum (**critical angle**). At the critical angle, the characteristic equation is

$$\text{TE) } \tan \left( \frac{\pi d \sqrt{\epsilon_{r1}} \cos \theta_c}{\lambda_{oc}} - \frac{m\pi}{2} \right) = \frac{\sqrt{\sin^2 \theta_c - \frac{\epsilon_2}{\epsilon_1}}}{\cos \theta_c} = 0$$

$$\text{TM) } \tan \left( \frac{\pi d \sqrt{\epsilon_{r1}} \cos \theta_c}{\lambda_{oc}} - \frac{m\pi}{2} \right) = \frac{\sqrt{\sin^2 \theta_c - \frac{\epsilon_2}{\epsilon_1}}}{\underbrace{(\epsilon_2 / \epsilon_1) \cos \theta_c}} = 0$$

$$\text{since } \theta_c = \sin^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

$$\text{for both TE and TM modes } \Rightarrow \frac{\pi d \sqrt{\epsilon_{r1}} \cos \theta_c}{\lambda_{oc}} = \frac{m\pi}{2}$$

The **cut-off wavelengths** (referenced to free space as usual in optical wave guides) and the corresponding **cut-off frequencies** for the guided modes are

$$\begin{aligned}\lambda_{oc} &= \frac{2d\sqrt{\epsilon_{r1}}}{m} \cos \theta_c = \frac{2d\sqrt{\epsilon_{r1}}}{m} \sqrt{1 - \sin^2 \theta_c} \\ &= \frac{2d\sqrt{\epsilon_{r1}}}{m} \sqrt{1 - \frac{\epsilon_{r2}}{\epsilon_{r1}}} = \frac{2d}{m} \sqrt{\epsilon_{r1} - \epsilon_{r2}}\end{aligned}$$

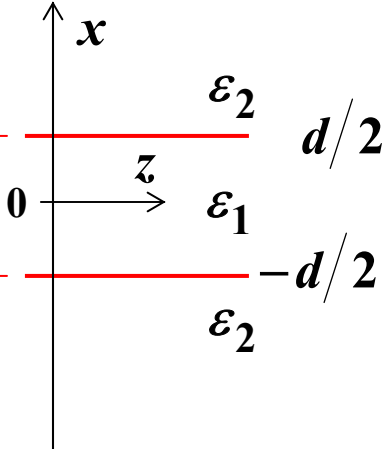
$$f = \frac{c}{\lambda_{oc}} = \frac{mc}{2d\sqrt{\epsilon_{r1} - \epsilon_{r2}}}, \quad m = 0, 1, 2, \dots$$

The **fundamental modes** are the **TE<sub>0</sub>** and the **TM<sub>0</sub>** with zero cut-off frequency.

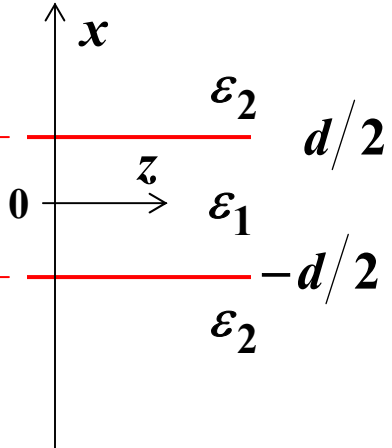
**TE** and **TM** modes with the **same index** form degenerate pairs with **identical** cut-off frequencies.

## Field expressions

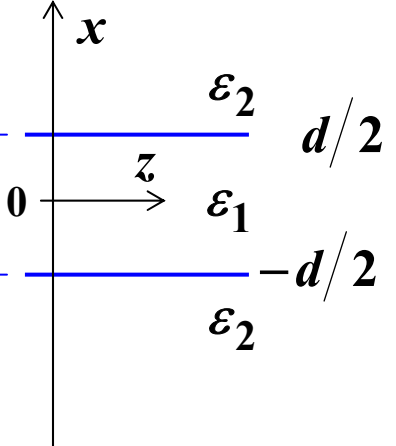
### Even TE modes

$$E_y = \begin{cases} E_o \cos(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ E_o \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ E_o \cos(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$


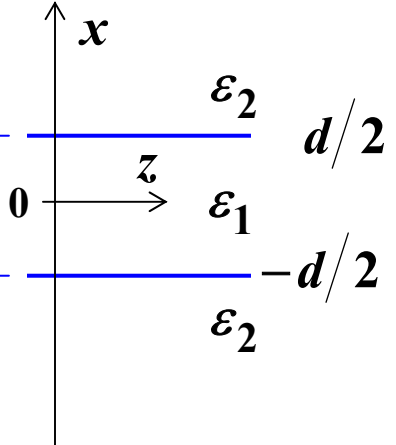
### Odd TE modes

$$E_y = \begin{cases} E_o \sin(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ E_o \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ E_o \sin(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$


## Even TM modes

$$H_y = \begin{cases} H_o \cos(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ H_o \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ H_o \cos(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$


## Odd TM modes

$$H_y = \begin{cases} H_o \sin(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ H_o \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ -H_o \sin(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$


**In medium 1**

$$\left(\beta_1\right)^2 = \beta_{x1}^2 + \beta_z^2 = \omega \sqrt{\mu_0 \epsilon_{r1} \epsilon_0}$$

**In medium 2**

$$\left(\beta_2\right)^2 = -\alpha_{x2}^2 + \beta_z^2 = \omega \sqrt{\mu_0 \epsilon_{r2} \epsilon_0}$$

**We have**

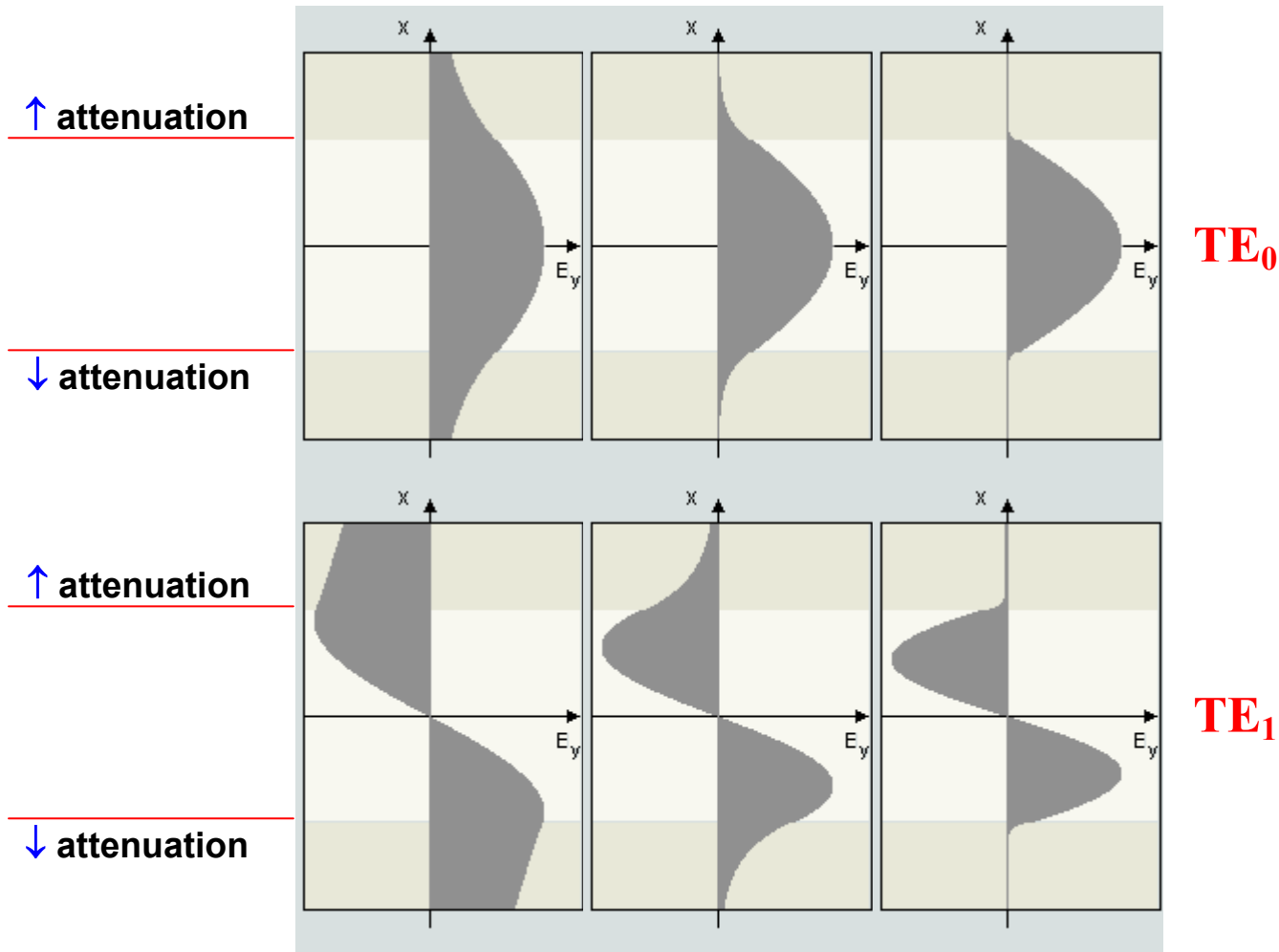
$$\beta_z = \beta_1 \sin \theta_i$$

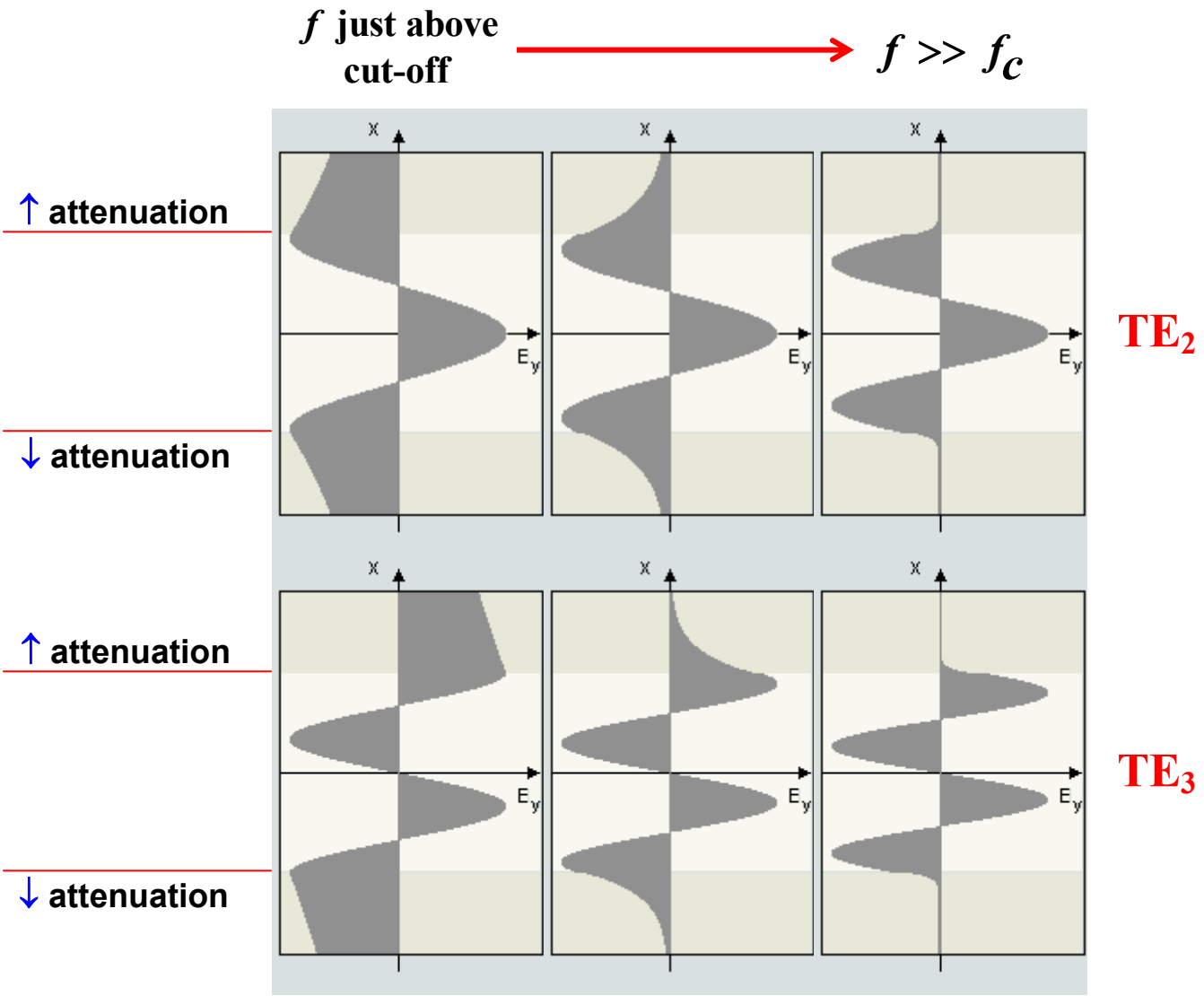
$$\beta_{x1} = \beta_1 \cos \theta_i$$

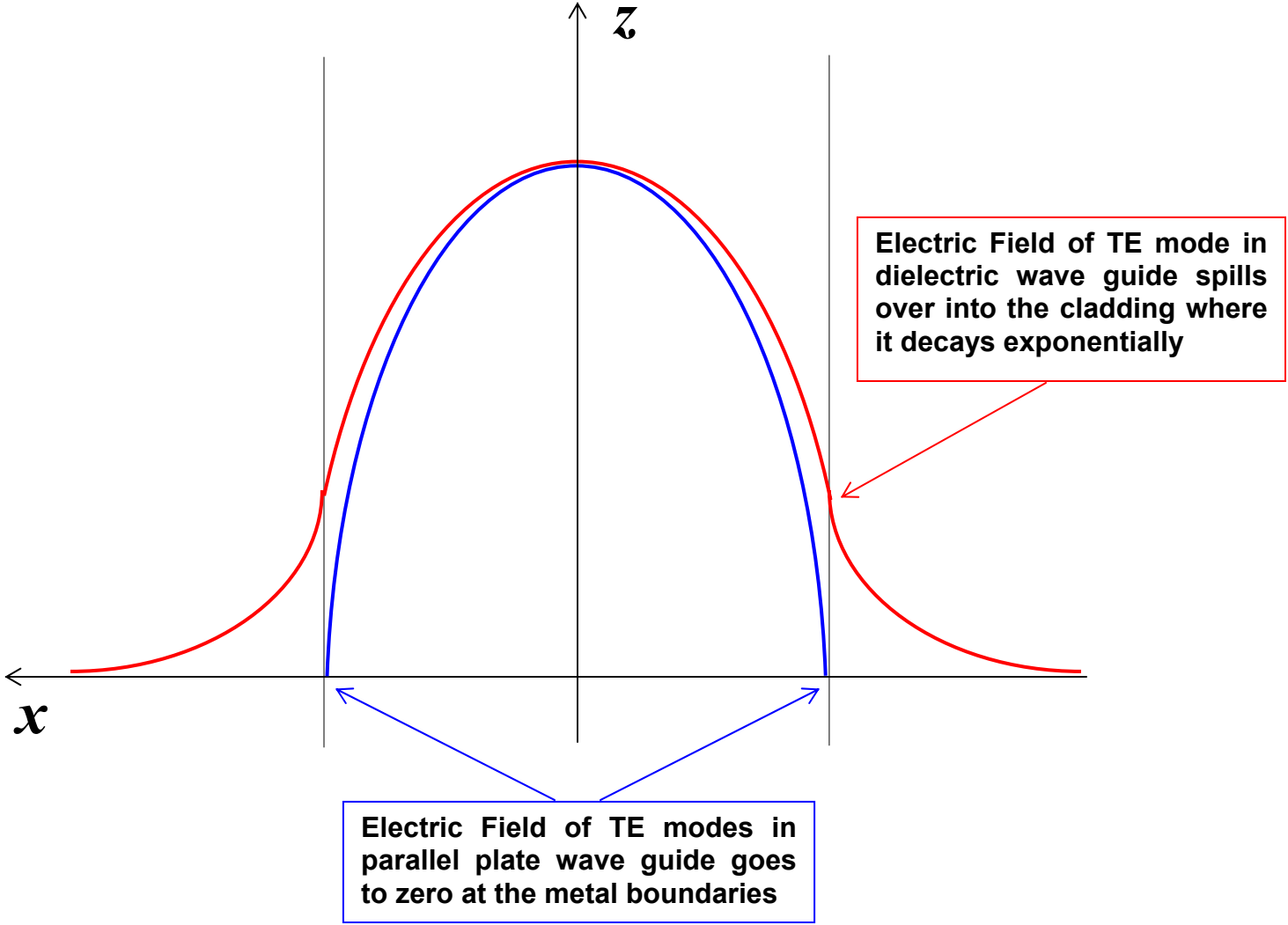
For each **mode** the angle of incidence is obtained from the solution of the **characteristic equation**.

Examples of profiles for the transverse **electric field** of **TE** modes.  
**TM** modes have similar profiles for the **magnetic field**.

$f$  just above cut-off  $\longrightarrow f \gg f_c$







**Magnetic field** components for **TE** modes are obtained from **Faraday's law**

$$\nabla \times \vec{\mathbf{E}} = -j\omega \mu \vec{\mathbf{H}}$$



$$\det \begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{E}_x = 0 & \mathbf{E}_y & \mathbf{E}_z = 0 \end{bmatrix} \Rightarrow \begin{aligned} -\frac{\partial}{\partial z} \mathbf{E}_y &= -j\omega \mu_0 \mathbf{H}_x \\ \frac{\partial}{\partial z} \mathbf{E}_x - \frac{\partial}{\partial x} \mathbf{E}_z &= -j\omega \mu_0 \mathbf{H}_y = 0 \\ \frac{\partial}{\partial x} \mathbf{E}_y &= -j\omega \mu_0 \mathbf{H}_z \end{aligned}$$

For example, the **transverse magnetic field** component is proportional to the (transverse) **electric field**. In the guide core:

$$\boxed{-\frac{\partial}{\partial z} \mathbf{E}_y = -j\omega \mu_0 \mathbf{H}_x}$$

$$\text{Even) } -\frac{\partial}{\partial z} E_o \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} = -j\omega \mu_0 \mathbf{H}_x$$

$$\text{Odd) } -\frac{\partial}{\partial z} E_o \sin(\beta_{x1} \cdot x) e^{-j\beta_z z}$$

$$\Rightarrow \mathbf{H}_x = -\frac{\beta_z}{\omega \mu_0} \mathbf{E}_y = \begin{cases} -\frac{\beta_z E_o}{\omega \mu_0} \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & \text{(Even)} \\ -\frac{\beta_z E_o}{\omega \mu_0} \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & \text{(Odd)} \end{cases}$$

**Electric field** components for **TM** modes are obtained from **Ampere's law**

$$\nabla \times \vec{\mathbf{H}} = j\omega \varepsilon \vec{\mathbf{E}}$$



$$\det \begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \cancel{\frac{\partial}{\partial y}} & \frac{\partial}{\partial z} \\ \mathbf{H}_x=0 & \mathbf{H}_y & \mathbf{H}_z=0 \end{bmatrix} \Rightarrow \begin{aligned} -\frac{\partial}{\partial z} \mathbf{H}_y &= j\omega \varepsilon \mathbf{E}_x \\ \frac{\partial}{\partial z} \mathbf{H}_x - \frac{\partial}{\partial x} \mathbf{H}_z &= 0 = j\omega \varepsilon \mathbf{E}_y \\ \frac{\partial}{\partial x} \mathbf{H}_y &= j\omega \varepsilon \mathbf{E}_z \end{aligned}$$

Next, is a summary of all the field components parallel to the plane of incidence.

## Even TE modes

$$H_x = \begin{cases} -\frac{\beta_z}{\omega \mu_0} E_0 \cos(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ -(\beta_z / \omega \mu_0) E_0 \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ -\frac{\beta_z}{\omega \mu_0} E_0 \cos(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$
  

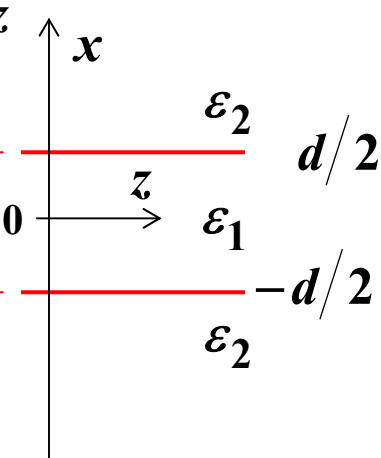
$$H_z = \begin{cases} -\frac{j\alpha_{x2}}{\omega \mu_0} E_0 \cos(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ -j(\beta_{x1} / \omega \mu_0) E_0 \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ \frac{j\alpha_{x2}}{\omega \mu_0} E_0 \cos(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$

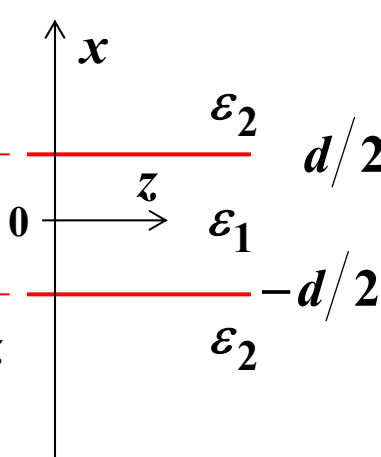
## Odd TE modes

$$H_x = \begin{cases} -\frac{\beta_z}{\omega \mu_0} E_0 \sin(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ -(\beta_z / \omega \mu_0) E_0 \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ \frac{\beta_z}{\omega \mu_0} E_0 \sin(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$

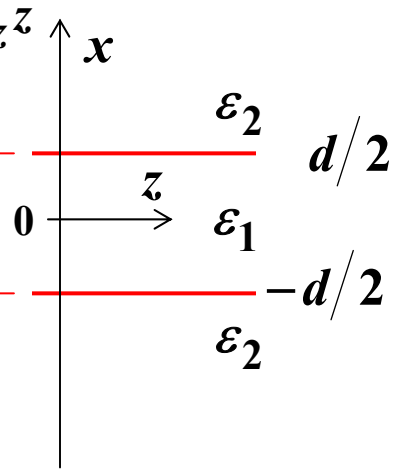
$$H_z = \begin{cases} -\frac{j\alpha_{x2}}{\omega \mu_0} E_0 \sin(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ j(\beta_{x1} / \omega \mu_0) E_0 \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ -\frac{j\alpha_{x2}}{\omega \mu_0} E_0 \sin(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$

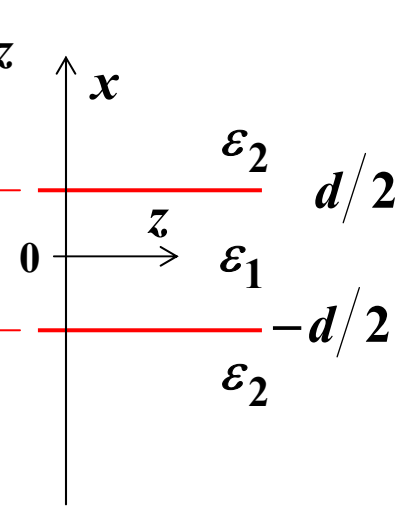
## Even TM modes

$$\mathbf{E}_x = \begin{cases} -\frac{\beta_z}{\omega \varepsilon_2} H_0 \cos(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ -(\beta_z / \omega \varepsilon_1) H_0 \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ -\frac{\beta_z}{\omega \varepsilon_2} H_0 \cos(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$

  

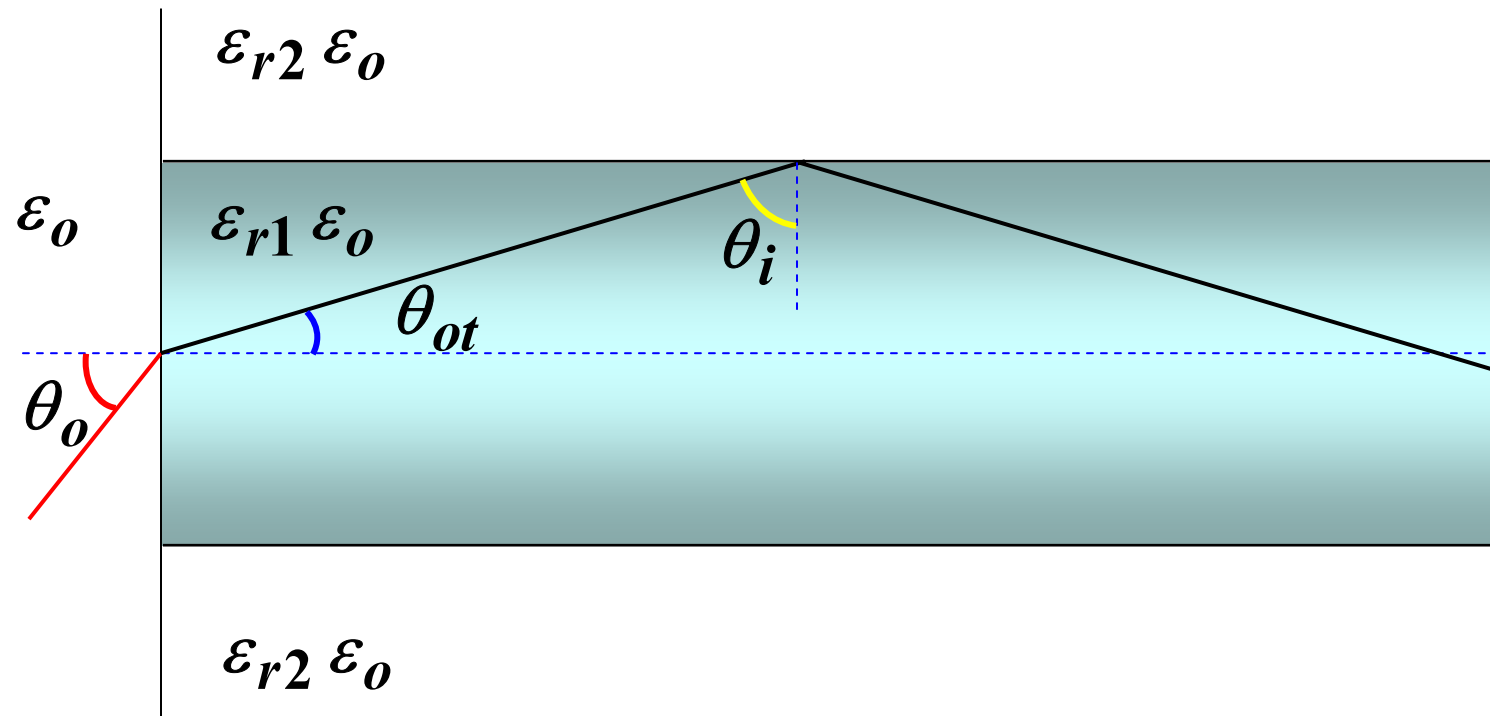
$$\mathbf{E}_z = \begin{cases} \frac{j\alpha_{x2}}{\omega \varepsilon_2} H_0 \cos(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ j(\beta_{x1} / \omega \varepsilon_1) H_0 \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ -\frac{j\alpha_{x2}}{\omega \varepsilon_2} H_0 \cos(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$


## Odd TM modes

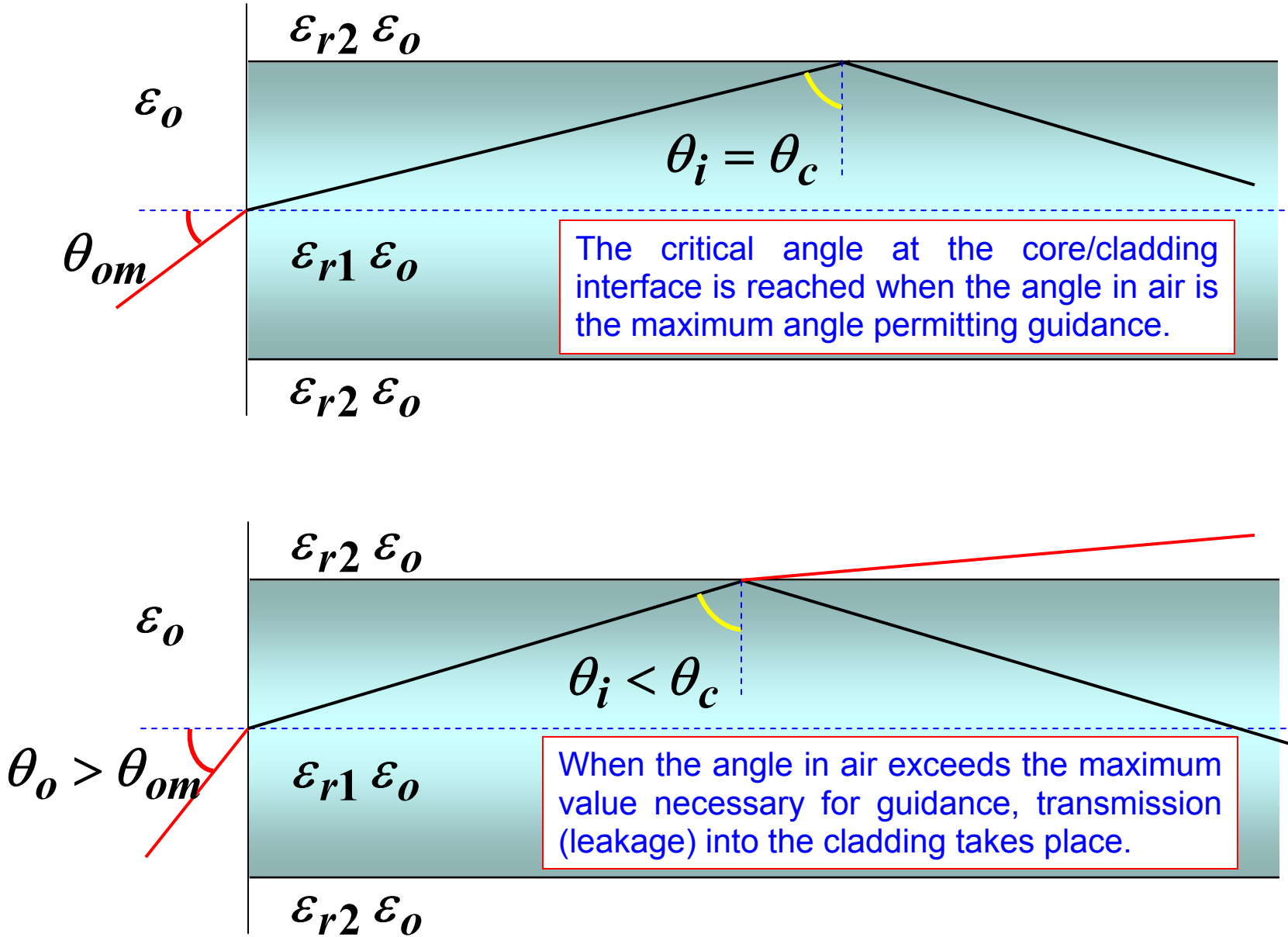
$$E_x = \begin{cases} -\frac{\beta_z}{\omega \epsilon_2} H_0 \sin(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ -(\beta_z / \omega \epsilon_1) H_0 \sin(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ \frac{\beta_z}{\omega \epsilon_2} H_0 \sin(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$

  

$$E_z = \begin{cases} \frac{j\alpha_{x2}}{\omega \epsilon_2} H_0 \sin(\beta_{x1} \cdot d/2) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} & x > d/2 \\ -j(\beta_{x1} / \omega \epsilon_1) H_0 \cos(\beta_{x1} \cdot x) e^{-j\beta_z z} & -d/2 < x < d/2 \\ \frac{j\alpha_{x2}}{\omega \epsilon_2} H_0 \sin(\beta_{x1} \cdot d/2) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} & x < -d/2 \end{cases}$$


Consider a wave entering the **end** of the wave guide from **air**.



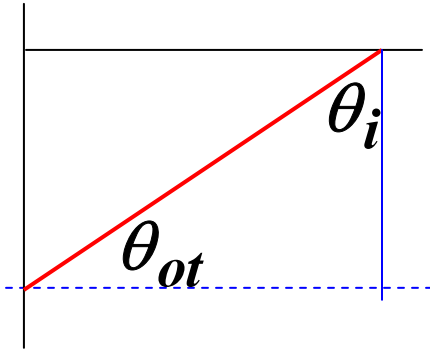
A classical problem of optical waveguides is to determine the **maximum angle of entrance**  $\theta_o$  that satisfies the condition of **total internal reflection** (guidance).



At the **air-core** interface

$$\sin \theta_{ot} = \sqrt{\frac{\epsilon_{rair} \epsilon_o}{\epsilon_{r1} \epsilon_o}} \sin \theta_o = \sqrt{\frac{1}{\epsilon_{r1}}} \sin \theta_o$$

$$\theta_i + \theta_{ot} = 90^\circ \Rightarrow \cos \theta_{ot} = \sin \theta_i \Leftarrow$$



At the **critical angle**

$$\sin \theta_i = \sin \theta_c = \sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

$$\sin^2 \theta_{otm} = 1 - \cos^2 \theta_{otm} = 1 - \sin^2 \theta_c = 1 - \frac{\epsilon_{r2}}{\epsilon_{r1}} = \frac{\sin^2 \theta_{om}}{\epsilon_{r1}}$$

$$\Rightarrow \sin \theta_{om} = \sqrt{\epsilon_{r1} - \epsilon_{r2}} = \text{numerical aperture}$$

$$\theta_{om} = \sin^{-1} \sqrt{\epsilon_{r1} - \epsilon_{r2}}$$