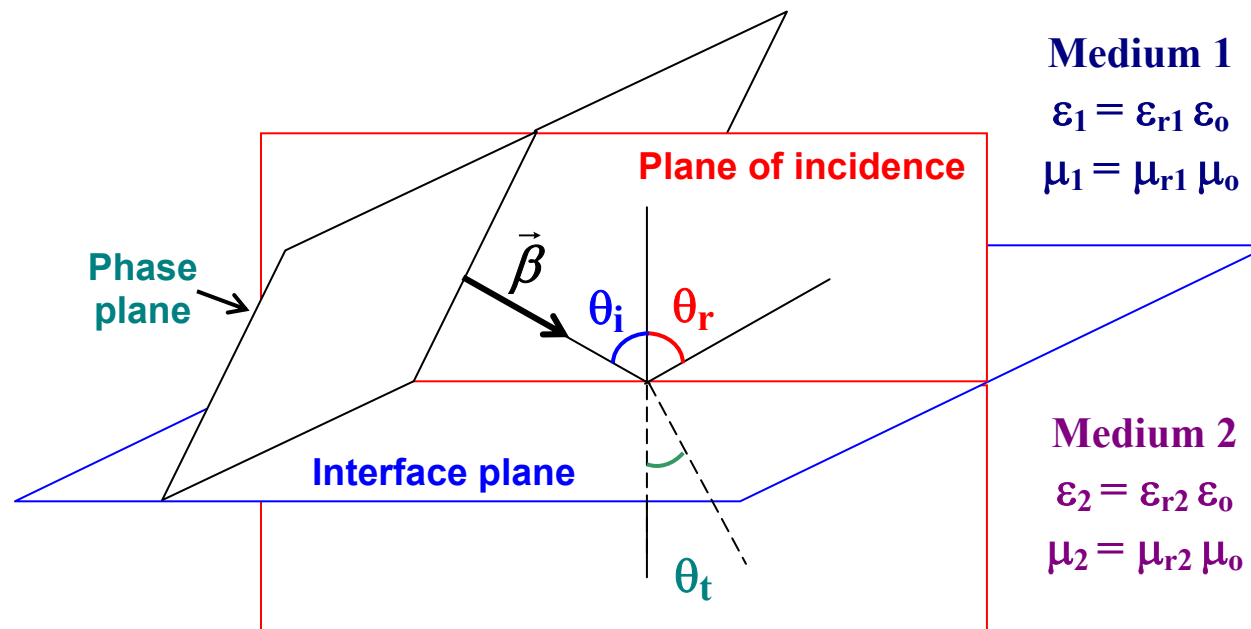


## Oblique incidence: Interface between dielectric media

Consider a **planar interface** between two **dielectric** media. A plane wave is incident at an angle from medium 1.

- The **interface plane** defines the boundary between the media.
- The **plane of incidence** contains the **propagation vector** and is both **perpendicular** to the **interface plane** and to the **phase planes** of the wave.



There are two elementary orientations (**polarizations**) for the electromagnetic fields:

- **Perpendicular Polarization**

The **electric field** is perpendicular to the plane of incidence and the **magnetic field** is parallel to the plane of incidence.

The fields are configured as in the Transverse Electric (**TE**) modes.

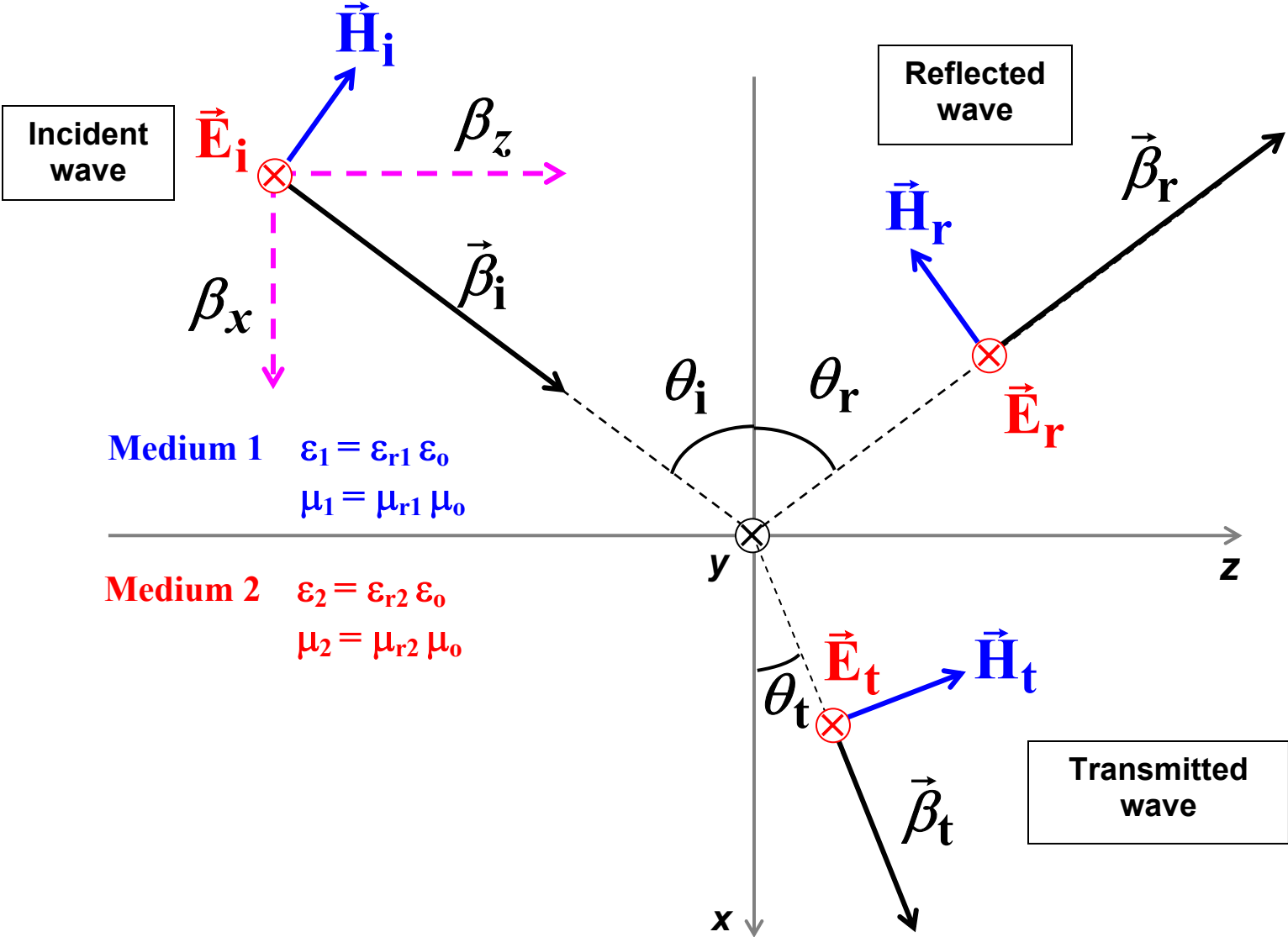
- **Parallel Polarization**

The **magnetic field** is perpendicular to the plane of incidence and the **electric field** is parallel to the plane of incidence.

The fields are configured as in the Transverse Magnetic (**TM**) modes.

**Any plane wave with general field orientation can be obtained by superposition of two waves with perpendicular and parallel polarization.**

### Perpendicular (TE) polarization



The **electric field** phasors for the **perpendicular polarization**, with reference to the system of coordinates in the figure, are given by

$$\vec{E}_i = E_{yi} e^{-j\beta_{ix} \cdot x - j\beta_{iz} \cdot z} \hat{i}_y$$

$$\vec{E}_r = E_{yr} e^{-j\beta_{rx} \cdot x - j\beta_{rz} \cdot z} \hat{i}_y$$

$$\vec{E}_t = E_{yt} e^{-j\beta_{tx} \cdot x - j\beta_{tz} \cdot z} \hat{i}_y$$

The **propagation vector** components in medium 1 are expressed as

$$|\vec{\beta}_i| = \sqrt{\beta_{ix}^2 + \beta_{iz}^2} = \beta_1 = \omega \sqrt{\mu_1 \epsilon_1}$$

$$\beta_{ix} = \beta_1 \cos \theta_i \quad \beta_{iz} = \beta_1 \sin \theta_i$$

$$|\vec{\beta}_r| = \sqrt{\beta_{rx}^2 + \beta_{rz}^2} = \beta_1$$

$$\beta_{rx} = -\beta_1 \cos \theta_r \quad \beta_{rz} = \beta_1 \sin \theta_r$$

The **propagation vector** components in medium 2 are expressed as

$$|\vec{\beta}_t| = \sqrt{\beta_{tx}^2 + \beta_{tz}^2} = \beta_2 = \omega \sqrt{\mu_2 \epsilon_2}$$

$$\beta_{tx} = \beta_2 \cos \theta_t \quad \beta_{tz} = \beta_2 \sin \theta_t$$

The **magnetic field** components can be obtained as

$$\vec{H}_i = \frac{\vec{\beta}_i \times \vec{E}_i}{\omega \mu_1} = \frac{E_{yi}}{\eta_1} (-\sin \theta_i \hat{i}_x + \cos \theta_i \hat{i}_z) e^{-j\beta_{ix}x - j\beta_{iz}z}$$

$$\vec{H}_r = \frac{\vec{\beta}_r \times \vec{E}_r}{\omega \mu_1} = -\frac{E_{yr}}{\eta_1} (\sin \theta_r \hat{i}_x + \cos \theta_r \hat{i}_z) e^{-j\beta_{rx}x - j\beta_{rz}z}$$

$$\vec{H}_t = \frac{\vec{\beta}_t \times \vec{E}_t}{\omega \mu_2} = \frac{E_{yt}}{\eta_2} (-\sin \theta_t \hat{i}_x + \cos \theta_t \hat{i}_z) e^{-j\beta_{tx}x - j\beta_{tz}z}$$

Assuming that the amplitude of the **incident** electric field is given, to completely specify the problem we need to find the **amplitude** of **reflected** and **transmitted** electric field.

The **boundary condition** at the interface ( $x = 0$ ) states that the **tangential electric field** must be continuous. Because of the **perpendicular polarization**, the tangential field is also the total field

$$x = 0) \quad E_{yi} e^{-j\beta_{iz}z} + E_{yr} e^{-j\beta_{rz}z} = E_{yt} e^{-j\beta_{tz}z}$$

The relation above must be valid for any choice of “z” and we must have (**phase conservation law**)

$$\beta_{iz} = \beta_{rz} = \beta_{tz}$$

The first equality indicates that the **reflected angle** is the **same** as the **incident angle**.

$$\beta_{iz} = \beta_{rz} \Rightarrow \beta_1 \sin \theta_i = \beta_1 \sin \theta_r \Rightarrow \theta_i = \theta_r$$

The second equality provides the **transmitted angle**

$$\beta_{iz} = \beta_{tz} \quad \Rightarrow \quad \beta_1 \sin \theta_i = \beta_2 \sin \theta_t$$

$$\Rightarrow \quad \theta_t = \sin^{-1} \left( \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin \theta_i \right) \quad \text{Snell's Law}$$

Since we have also

$$e^{-j\beta_{iz}z} = e^{-j\beta_{rz}z} = e^{-j\beta_{tz}z}$$

the **boundary condition** for the **electric field** becomes

$$E_{yi} + E_{yr} = E_{yt}$$

The **tangential magnetic field** must also be continuous at the interface. This applies in our case to the z-components

$$H_{zi} + H_{zr} = H_{zt}$$

$$\frac{E_{yi}}{\eta_1} \cos \theta_i - \frac{E_{yr}}{\eta_1} \cos \theta_i = \frac{E_{yt}}{\eta_2} \cos \theta_t$$

$$\Rightarrow E_{yi} - E_{yr} = \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} E_{yt}$$

Solution of the system of **boundary equations** gives

$$\Gamma_{\perp}(E) = \frac{E_{yr}}{E_{yi}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \quad \text{Reflection coefficient}$$

$$\tau_{\perp}(E) = \frac{E_{yt}}{E_{yi}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \quad \text{Transmission coefficient}$$

For the **magnetic field**, we can define the **reflection coefficient** as

$$\Gamma_{\perp}(H) = \frac{H_{zr}}{H_{zi}} = -\frac{H_r}{H_i}$$

In terms of **electric field**, the magnetic field components are

$$H_{zr} = \frac{-E_{yr}}{\eta_1} \cos \theta_i = -H_r \cos \theta_i$$

$$H_{zi} = \frac{E_{yi}}{\eta_1} \cos \theta_i = H_i \cos \theta_i$$

The **reflection coefficient** for the magnetic field is then

$$\Gamma_{\perp}(H) = \frac{-E_{yr}}{E_{yi}} = -\Gamma_{\perp}(E) = \frac{\eta_1 \cos \theta_t - \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

The **transmission coefficient** is defined as

$$\tau_{\perp}(H) = \frac{H_t}{H_i}$$

The magnetic field components are

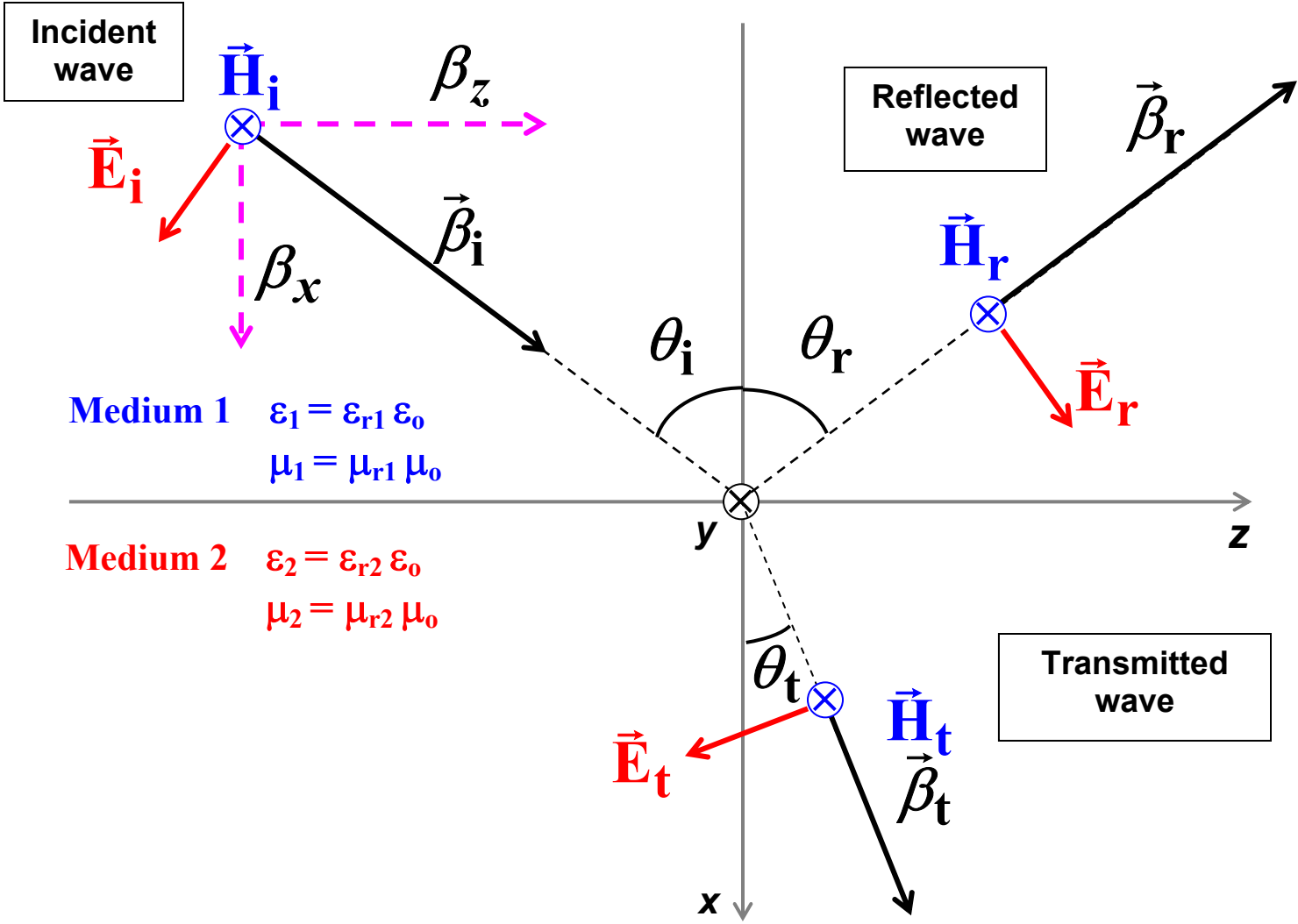
$$H_t = \frac{E_{yt}}{\eta_2}$$

$$H_i = \frac{E_{yi}}{\eta_1}$$

The **transmission coefficient** for the magnetic field is then

$$\tau_{\perp}(H) = \frac{H_t}{H_i} = \frac{\eta_1}{\eta_2} \tau_{\perp}(E) = \frac{2\eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

# Parallel (TM) polarization



The **magnetic field** phasors for the **parallel polarization** are given by

$$\vec{H}_i = H_{yi} e^{-j\beta_{ix} \cdot x - j\beta_{iz} \cdot z} \hat{i}_y$$

$$\vec{H}_r = H_{yr} e^{-j\beta_{rx} \cdot x - j\beta_{rz} \cdot z} \hat{i}_y$$

$$\vec{H}_t = H_{yt} e^{-j\beta_{tx} \cdot x - j\beta_{tz} \cdot z} \hat{i}_y$$

and the **electric field** components can be obtained as

$$\vec{E}_i = -\frac{\vec{\beta}_i \times \vec{H}_i}{\omega \epsilon_1} = \eta_1 H_{yi} (\sin \theta_i \hat{i}_x - \cos \theta_i \hat{i}_z) e^{-j\beta_{ix} x - j\beta_{iz} z}$$

$$\vec{E}_r = -\frac{\vec{\beta}_r \times \vec{H}_r}{\omega \epsilon_1} = \eta_1 H_{yr} (\sin \theta_r \hat{i}_x + \cos \theta_r \hat{i}_z) e^{-j\beta_{rx} x - j\beta_{rz} z}$$

$$\vec{E}_t = -\frac{\vec{\beta}_t \times \vec{H}_t}{\omega \epsilon_2} = \eta_2 H_{yt} (\sin \theta_t \hat{i}_x - \cos \theta_t \hat{i}_z) e^{-j\beta_{tx} x - j\beta_{tz} z}$$

Also for **parallel polarization** one can verify that the same relationships between angles apply, as found earlier for the perpendicular polarization, including **Snell's law**

$$\theta_i = \theta_r$$

$$\theta_t = \sin^{-1} \left( \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin \theta_i \right)$$

We have again two boundary conditions at the interface. One condition is for continuity of the **tangential magnetic field**

$$H_{yi} + H_{yr} = H_{yt}$$

A second condition is for continuity of the **tangential electric field**

$$E_{zi} + E_{zr} = E_{zt}$$

$$-\eta_1 \cos \theta_i H_{yi} + \eta_1 \cos \theta_i H_{yr} = -\eta_2 \cos \theta_t H_{yt}$$

$$\Rightarrow H_{yi} - H_{yr} = \frac{\eta_2 \cos \theta_t}{\eta_1 \cos \theta_i} H_{yt}$$

From the equations provided by the **boundary conditions** we obtain the **reflection** and **transmission** coefficients for the magnetic field of a wave with parallel polarization as

$$\Gamma_{\parallel}(H) = \frac{H_{yr}}{H_{yi}} = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_t}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t}$$

$$\tau_{\parallel}(H) = \frac{H_{yt}}{H_{yi}} = \frac{2\eta_1 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t}$$

The **reflection coefficient** for the electric field is defined as

$$\Gamma_{\parallel}(E) = \frac{E_{zr}}{E_{zi}} = -\frac{E_r}{E_i}$$

The tangential components of the electric field can be expressed in terms of magnetic field as

$$E_{zr} = E_r \cos \theta_i = \eta_1 \cos \theta_i H_{yr}$$

$$E_{zi} = -E_i \cos \theta_i = -\eta_1 \cos \theta_i H_{yi}$$

The reflection coefficient for the electric field is

$$\Gamma_{\parallel}(E) = \frac{E_{zr}}{E_{zi}} = -\frac{H_{yr}}{H_{yi}} = -\Gamma_{\parallel}(H) = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t}$$

The **transmission coefficient** for the electric field is defined as

$$\tau_{\parallel}(E) = \frac{E_t}{E_i}$$

The electric field components are given by

$$E_t = \eta_2 H_{yt}$$

$$E_i = \eta_1 H_{yi}$$

The transmission coefficient for the electric field becomes

$$\begin{aligned} \tau_{\parallel}(E) &= \frac{E_t}{E_i} = \frac{\eta_2 H_{yt}}{\eta_1 H_{yi}} = \frac{\eta_2}{\eta_1} \tau_{\parallel}(H) \\ &= \frac{\eta_2}{\eta_1} \frac{2\eta_1 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t} = \frac{2\eta_2 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t} \end{aligned}$$

Considerable simplifications are possible for the common case of **nonmagnetic dielectric media** with

$$\mu_1 = \mu_2 = \mu_0$$

First of all, **Snell's law** becomes

$$\theta_t = \sin^{-1} \left( \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin \theta_i \right) = \sin^{-1} \left( \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin \theta_i \right)$$

or, equivalently

$$\frac{\sin \theta_i}{\sin \theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \quad (n = \text{index of refraction})$$

Snell's law provides then a useful recipe to express the reflection and transmission coefficients only with **angles**, thus eliminating the explicit dependence on medium impedance.

After some trigonometric manipulations, we obtain the following table of **simplified coefficients** for electric and magnetic field

$$\Gamma_{\perp}(E) = -\Gamma_{\perp}(H) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$\Gamma_{\parallel}(E) = -\Gamma_{\parallel}(H) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$\tau_{\perp}(E) = \tau_{\perp}(H) \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t)}$$

$$\tau_{\parallel}(E) = \tau_{\parallel}(H) \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)}$$

## Power flow

The **time-average power flow normal** to the interface must be **continuous**. We can express this as

$$\frac{1}{2} \frac{E_i^2}{\eta_1} \cos \theta_i - \frac{1}{2} \frac{E_r^2}{\eta_1} \cos \theta_i = \frac{1}{2} \frac{E_t^2}{\eta_2} \cos \theta_t$$

**incident power** – **reflected power** = **transmitted power**

We define the reflection and transmission coefficients for the time-average power as

$$R = \frac{\text{reflected power}}{\text{incident power}} = \frac{E_r^2}{E_i^2}$$

$$T = \frac{\text{transmitted power}}{\text{incident power}} = \frac{E_t^2}{E_i^2} \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i}$$

The following **conversion formulas** relate **power** and **electric field coefficients**

$$R = \left| \Gamma^2(E) \right|$$

$$T = \left| \tau^2(E) \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} \right|$$

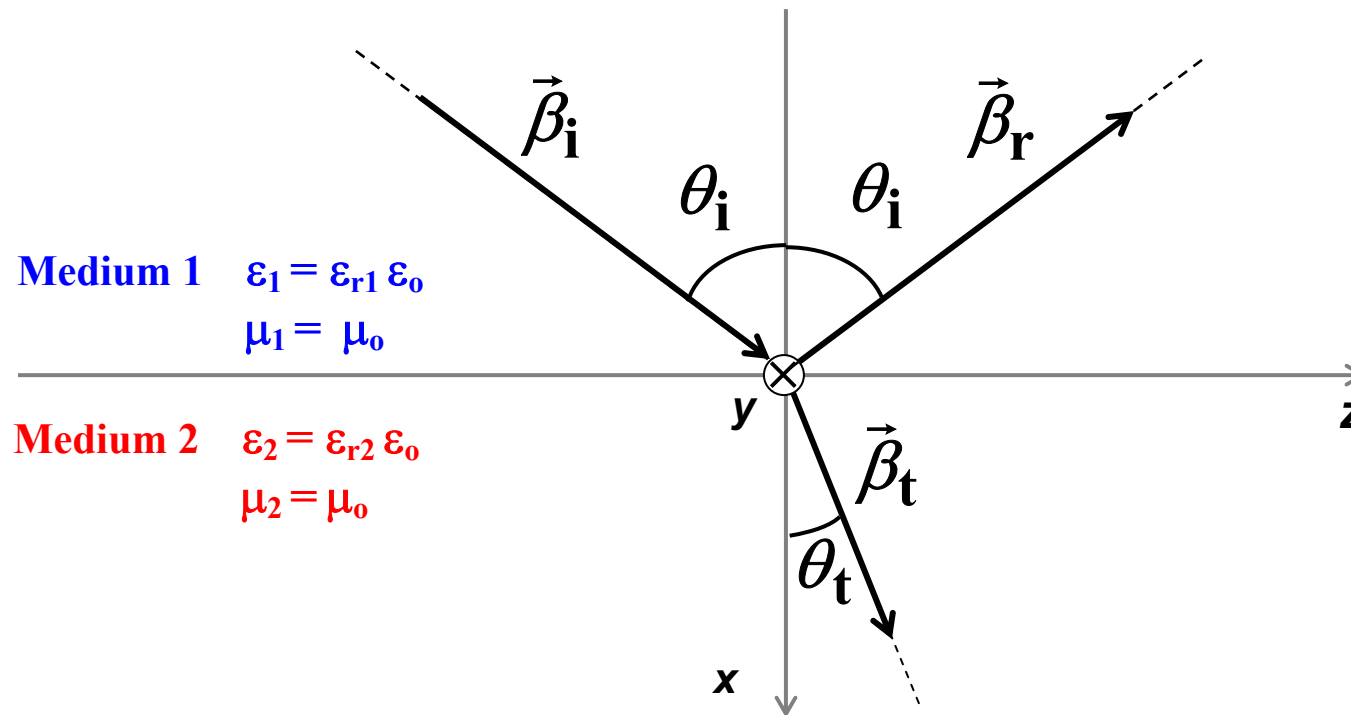
Note that **reflection** and **transmission** coefficients for the time-average power are always **real positive quantities**. The following power conservation condition is always verified

$$R + T = 1$$

Since the power flow normal to the interface is considered, the results obtained above apply equally to perpendicular and parallel polarization.

## Non-magnetic perfect dielectric media

*Case  $\epsilon_2 > \epsilon_1$*



**From Snell's law**

$$\sqrt{\epsilon_1} \sin \theta_i = \sqrt{\epsilon_2} \sin \theta_t \quad \epsilon_2 > \epsilon_1 \Rightarrow \theta_t < \theta_i$$

Since  $\theta_t < \theta_i$  there is always a transmitted (refracted) beam.

The transmission coefficients are always positive

$$\tau_{\perp}(E) = \tau_{\perp}(H) \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t)} > 0$$

$$\tau_{\parallel}(E) = \tau_{\parallel}(H) \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} > 0$$

$\Rightarrow$  transmitted and incident wave are in phase at the boundary.

$$\tau_{\perp}(E) = \frac{E_{yt}}{E_{yi}} \quad \frac{\begin{array}{c} E_{yi} \otimes \longrightarrow \\ H_{zi} \end{array}}{\begin{array}{c} E_{yt} \otimes \longrightarrow \\ H_{zt} \end{array}}$$

## Perpendicular polarization

$$\Gamma_{\perp}(E) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \quad \Gamma_{\perp}(H) = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

- The **reflection coefficient** for the **electric field** is always **negative**

$E_{yi}$  and  $E_{yr}$  have always phase difference of  $180^\circ$

- The **reflection coefficient** for the **magnetic field** is always **positive**

$H_{zi}$  and  $H_{zr}$  are always in phase

$$\Gamma_{\perp}(E) = \frac{E_{yr}}{E_{yi}} \quad \Gamma_{\perp}(H) = \frac{H_{zr}}{H_{zi}} \quad \begin{array}{c} E_{yi} \quad H_{zi} \quad E_{yr} \quad H_{zr} \\ \otimes \longrightarrow \quad \odot \longrightarrow \end{array}$$

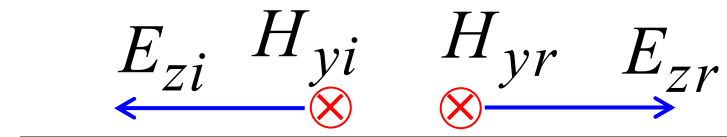
**Parallel polarization**

$$\Gamma_{\parallel}(E) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \quad \Gamma_{\parallel}(H) = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

When  $\theta_i + \theta_t < 90^\circ \Rightarrow \tan(\theta_i + \theta_t) > 0$

$E_{zi}$  and  $E_{zr}$  have phase difference of  $180^\circ$

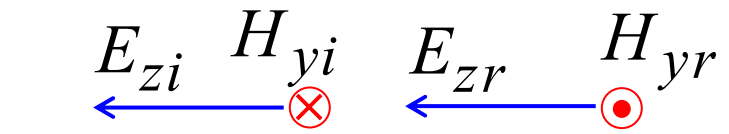
$H_{yi}$  and  $H_{yr}$  are in phase



When  $\theta_i + \theta_t > 90^\circ \Rightarrow \tan(\theta_i + \theta_t) < 0$

$E_{zi}$  and  $E_{zr}$  are in phase

$H_{yi}$  and  $H_{yr}$  have phase difference of  $180^\circ$



When  $\theta_i + \theta_t = 90^\circ \Rightarrow \tan(\theta_i + \theta_t) \rightarrow \infty$

the reflection coefficients vanish (**TOTAL TRANSMISSION**)

$$\Gamma_{\parallel}(E) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \rightarrow 0; \quad \Gamma_{\parallel}(H) = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \rightarrow 0$$

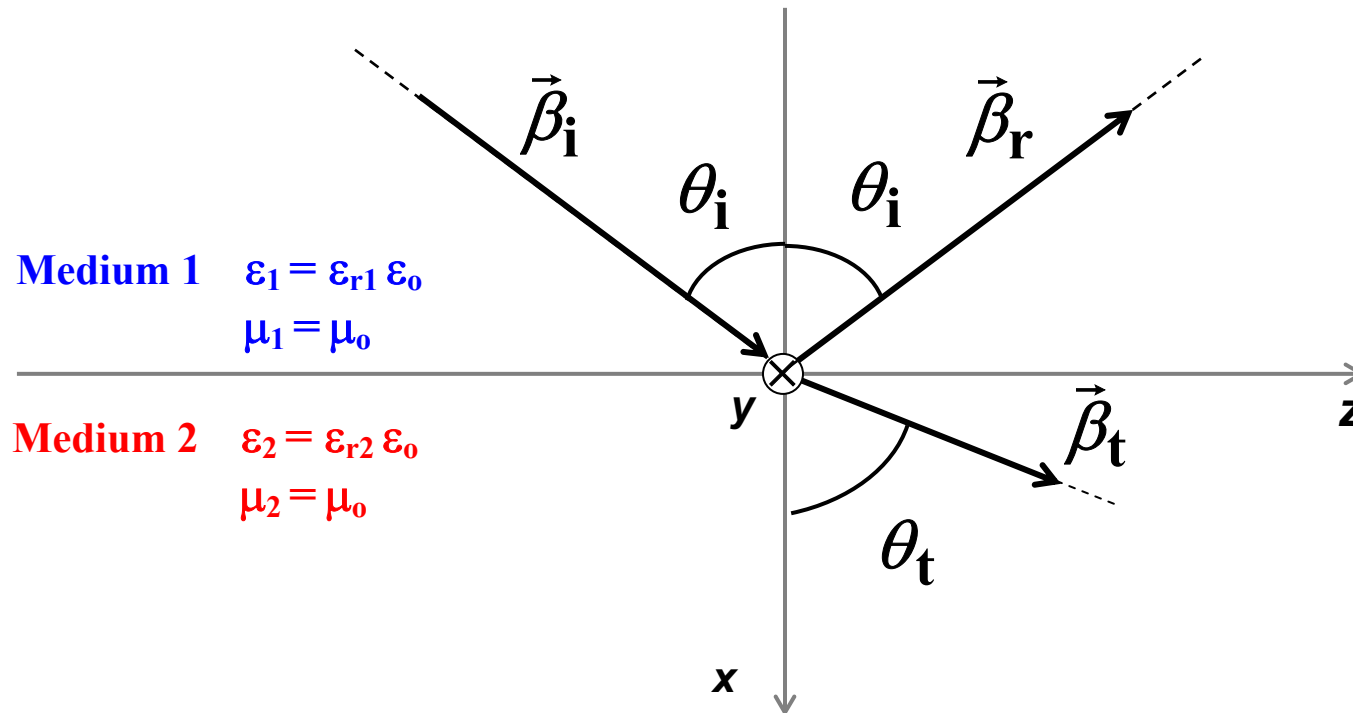
For  $\theta_i + \theta_t = 90^\circ \Rightarrow \theta_i = \theta_B$  (Brewster angle)

$$\text{and } \Rightarrow \cos \theta_B = \sin \theta_t$$

**From Snell's law**

$$\frac{\sin \theta_B}{\sin \theta_t} = \frac{\sin \theta_B}{\cos \theta_B} = \tan \theta_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \Rightarrow \theta_B = \tan^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

**Case  $\epsilon_2 < \epsilon_1$**



**From Snell's law**

$$\sqrt{\epsilon_1} \sin \theta_i = \sqrt{\epsilon_2} \sin \theta_t \quad \epsilon_2 < \epsilon_1 \Rightarrow \theta_t > \theta_i$$

For angles of incidence such that

$$\sin \theta_t < 1$$

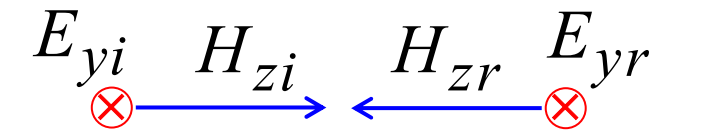
we have for **perpendicular polarization**

$$\Gamma_{\perp}(E) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_i + \theta_t)} > 0$$

$E_{yi}$  and  $E_{yr}$  are always in phase

$$\Gamma_{\perp}(H) = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = -\frac{\sin(\theta_t - \theta_i)}{\sin(\theta_i + \theta_t)} < 0$$

$H_{zi}$  and  $H_{zr}$  have always phase difference of  $180^\circ$

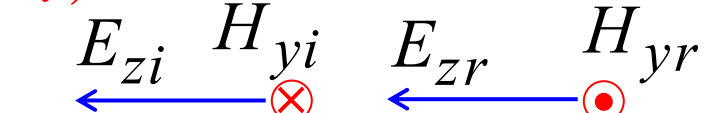


**For parallel polarization**

$$\Gamma_{\parallel}(E) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)}$$

$$\Gamma_{\parallel}(H) = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = -\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)}$$

When  $\theta_i + \theta_t < 90^\circ \Rightarrow \tan(\theta_i + \theta_t) > 0$

$E_{zi}$  and  $E_{zr}$  are in phase 

$H_{yi}$  and  $H_{yr}$  have phase difference of  $180^\circ$

When  $\theta_i + \theta_t > 90^\circ \Rightarrow \tan(\theta_i + \theta_t) < 0$

$E_{zi}$  and  $E_{zr}$  have phase difference of  $180^\circ$

$H_{yi}$  and  $H_{yr}$  are in phase 

When  $\theta_i + \theta_t = 90^\circ \Rightarrow \tan(\theta_i + \theta_t) \rightarrow \infty$

Also in this case the reflection coefficients vanish and we have **TOTAL TRANSMISSION**

$$\Gamma_{\parallel}(E) = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)} \rightarrow 0; \quad \Gamma_{\parallel}(H) = -\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)} \rightarrow 0$$

Total transmission occurs again, for **parallel polarization** only, at the **Brewster angle**

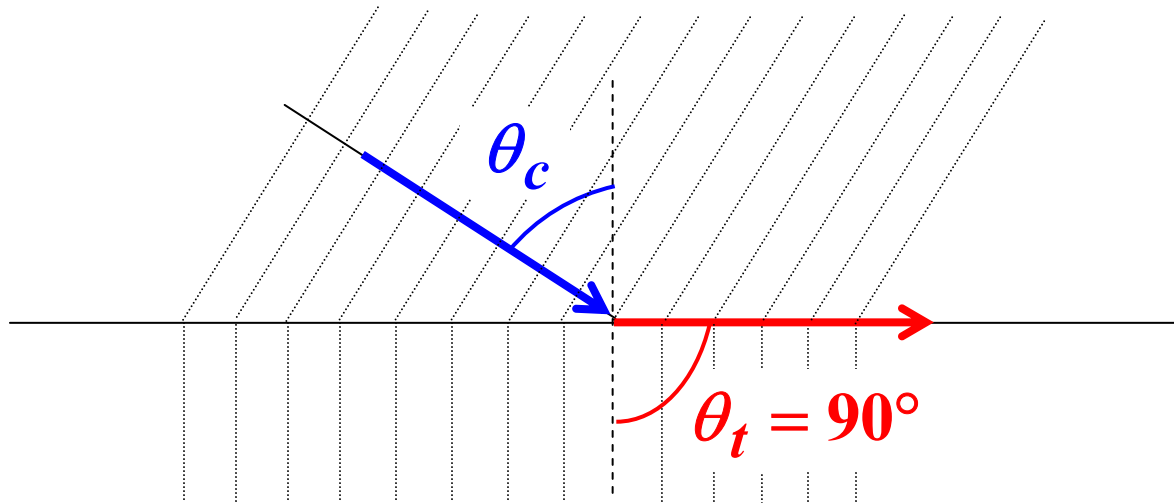
$$\theta_i = \theta_B = \tan^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

When

$$\sin \theta_i = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \sin \theta_t = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \Rightarrow \sin \theta_t = 1 \Rightarrow \theta_t = 90^\circ$$

we have a limit condition for **TOTAL REFLECTION**, valid for both polarizations. This particular angle of incidence is called

**critical angle**  $\theta_i = \theta_c = \sin^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}}$



For angles of incidence **beyond** the **critical angle**

$$\theta_i > \theta_c \Rightarrow \sin \theta_t = \sin \theta_i \sqrt{\frac{\epsilon_1}{\epsilon_2}} > 1$$

$$\Rightarrow \cos \theta_t = \text{imaginary}$$

$\pm$ , choose "-"

$$\cos \theta_t = -\sqrt{1 - \sin^2 \theta_t} = -j \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sqrt{\sin^2 \theta_i - \frac{\epsilon_2}{\epsilon_1}}$$

The **negative sign** is selected, in order to obtain the proper wave vector in medium 2, as shown later.

The **reflection** and **transmission** coefficients become **complex**

$$\Gamma_{\perp}(E) = -\Gamma_{\perp}(H) = \frac{\cos \theta_i + j \sqrt{\sin^2 \theta_i - \varepsilon_2 / \varepsilon_1}}{\cos \theta_i - j \sqrt{\sin^2 \theta_i - \varepsilon_2 / \varepsilon_1}}$$

$$\Gamma_{\parallel}(E) = -\Gamma_{\parallel}(H) = \frac{-\frac{\varepsilon_2}{\varepsilon_1} \cos \theta_i - j \sqrt{\sin^2 \theta_i - \varepsilon_2 / \varepsilon_1}}{\frac{\varepsilon_2}{\varepsilon_1} \cos \theta_i - j \sqrt{\sin^2 \theta_i - \varepsilon_2 / \varepsilon_1}}$$

$$\tau_{\perp}(E) = \tau_{\perp}(H) \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2 \cos \theta_i}{\cos \theta_i - j \sqrt{\sin^2 \theta_i - \varepsilon_2 / \varepsilon_1}}$$

$$\tau_{\parallel}(E) = \tau_{\parallel}(H) \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2 \sqrt{\varepsilon_2 / \varepsilon_1} \cos \theta_i}{\frac{\varepsilon_2}{\varepsilon_1} \cos \theta_i - j \sqrt{\sin^2 \theta_i - \varepsilon_2 / \varepsilon_1}}$$

If we consider the coefficients for **time-average power flow**, we have, **for both polarizations**

$$R = \Gamma(E) \cdot \Gamma^*(E) = 1$$

$$T = 1 - R = 0$$

This means that incident and reflected waves carry the same time-average power, and no power is transmitted to medium 2. But this does not mean that the field disappears in medium 2. The instantaneous power that enters medium 2 is eventually reflected back to medium 1.

The **electric field phasor** of the transmitted wave has the form

$$E_t = E_t e^{-j\beta_t x \cdot x} e^{-j\beta_t z \cdot z} = E_t e^{-j\beta_2 \cos \theta_t \cdot x} e^{-j\beta_2 \sin \theta_t \cdot z}$$

The **wave vector** components are

$$\begin{aligned}\beta_2 \cos \theta_t &= \omega \sqrt{\mu_0 \varepsilon_2} \left( -j \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sqrt{\sin^2 \theta_i - \frac{\varepsilon_2}{\varepsilon_1}} \right) \\ &= -j \omega \sqrt{\mu_0 \varepsilon_1} \sqrt{\sin^2 \theta_i - \frac{\varepsilon_2}{\varepsilon_1}} = -j \beta_1 \sqrt{\sin^2 \theta_i - \frac{\varepsilon_2}{\varepsilon_1}} = -j \alpha_t \\ \beta_2 \sin \theta_t &= \omega \sqrt{\mu_0 \varepsilon_2} \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin \theta_i = \beta_1 \sin \theta_i = \beta_{iz}\end{aligned}$$

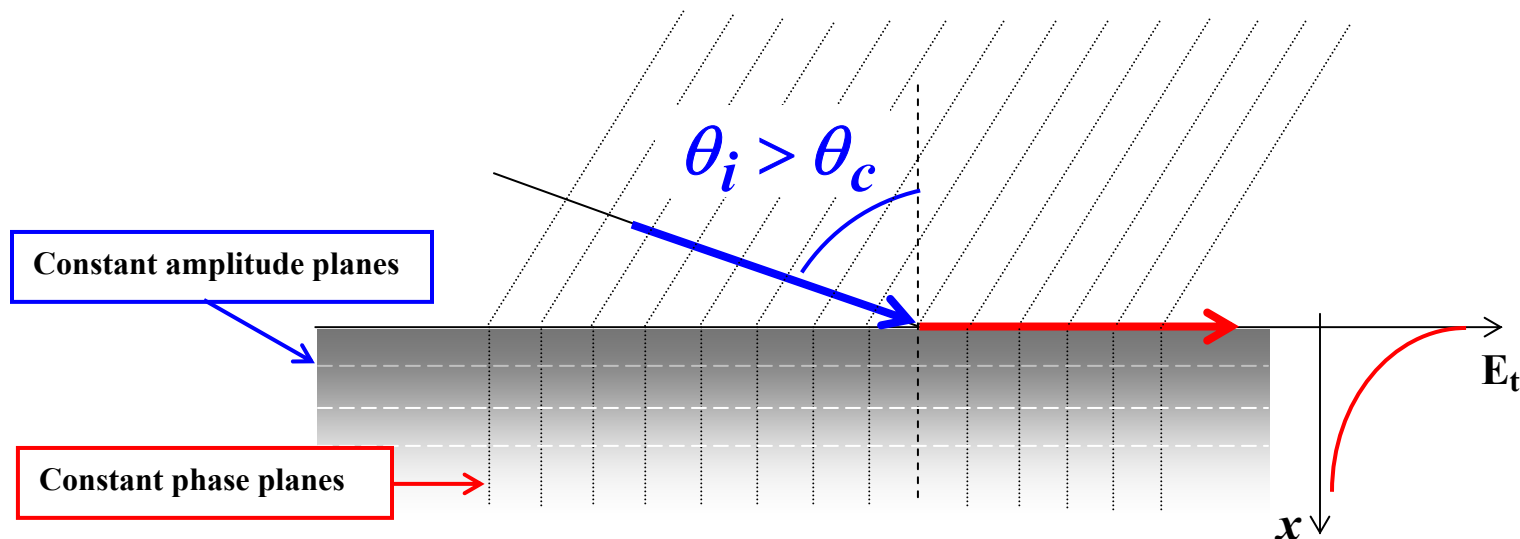
The field in **medium 2** corresponds to a **surface wave**, moving along the z-direction and **exponentially decaying (evanescent)** along the x-direction

$$E_t = E_t e^{-j(-j\alpha_t \cdot x)} e^{-j\beta_{iz} \cdot z} = E_t e^{-\alpha_t \cdot x} e^{-j\beta_{iz} \cdot z}$$

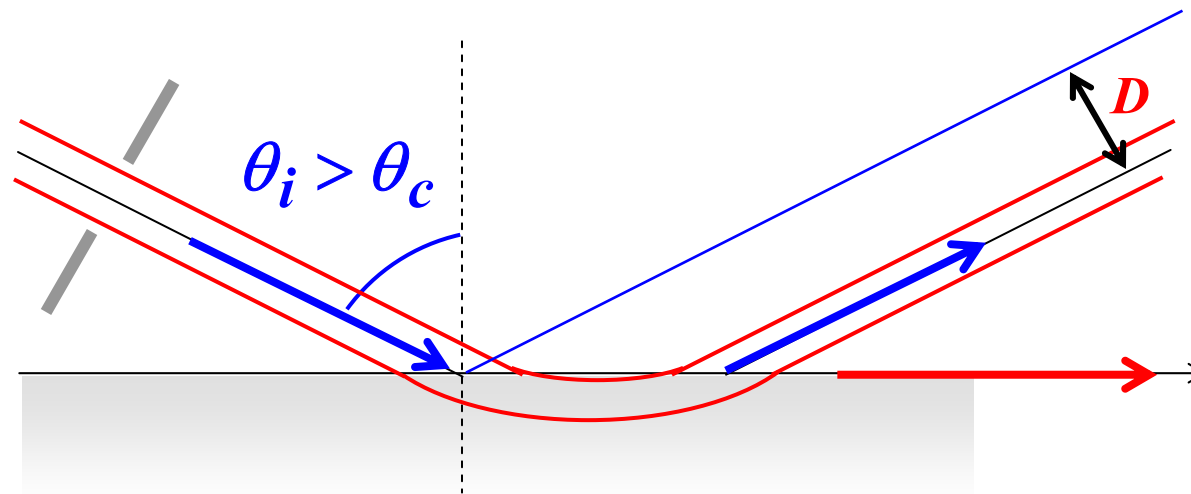
The **surface wave** moves **parallel** to the surface, with a phase velocity equal to the apparent phase velocity along  $z$  of the incident wave in medium 1

$$v_{p2} = \frac{v_{p1}}{\sin \theta_i} = v_{pz} > v_{p1}$$

For the surface wave, planes of **constant amplitude** are **parallel** and planes of **constant phase** are **normal** to the interface. These planes do not coincide, therefore the **surface wave** is a **nontransverse** wave.



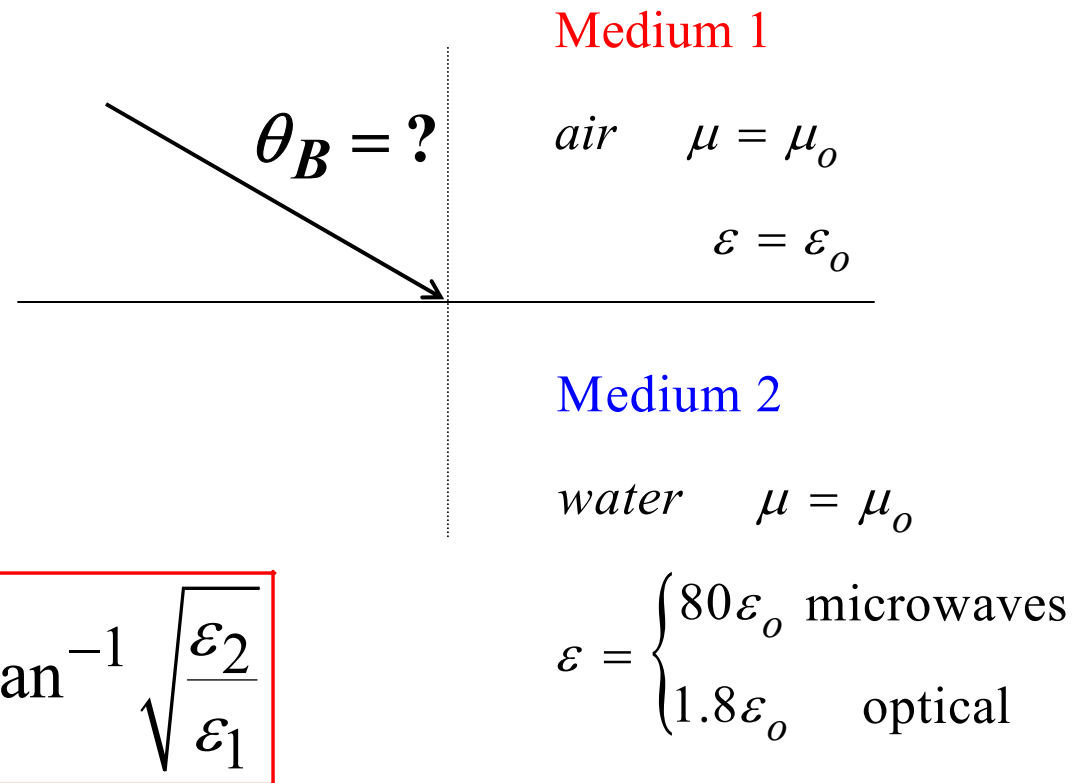
If you consider a **beam** incident on the interface, it is found that the power is totally reflected but after penetrating for some distance into medium 2. The reflected beam emerges displaced by a distance  $D$  (called **Goos-Hänchen shift**, discovered in 1947)



From experiments, the displacement is found to be

$$D \approx 0.52\sqrt{\varepsilon_2} \frac{2\pi}{\beta_1 \sqrt{\sin^2 \theta_i - \frac{\varepsilon_2}{\varepsilon_1}}} = 0.52\sqrt{\varepsilon_2} \frac{2\pi}{\alpha_t}$$

## Examples:



$$\theta_B = \tan^{-1} \sqrt{80} \cong 83.6^\circ \quad \text{microwaves}$$

$$\theta_B = \tan^{-1} \sqrt{1.8} \cong 53.3^\circ \quad \text{optical}$$

## At the Brewster angle

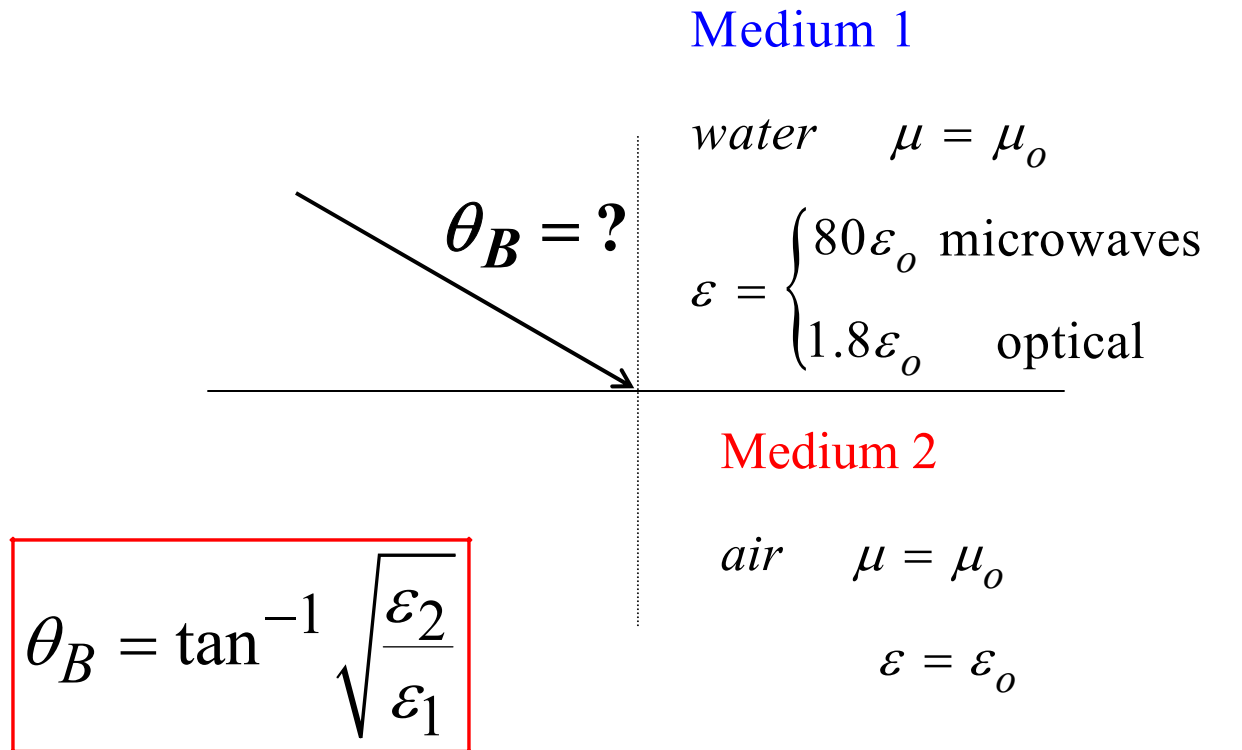
$$\theta_i + \theta_t = \theta_B + \theta_t = 90^\circ$$

$$\Rightarrow \theta_t \cong \begin{cases} 6.38^\circ & \text{microwaves} \\ 36.7^\circ & \text{optical} \end{cases}$$

## Verification with Snell's law

$$\sqrt{\varepsilon_1} \sin \theta_B = \sqrt{\varepsilon_2} \sin \theta_t$$

$$\theta_t = \sin^{-1} \left( \frac{\sin \theta_B}{\sqrt{\varepsilon_2}} \right) \cong \begin{cases} 6.38^\circ & \text{microwaves} \\ 36.7^\circ & \text{optical} \end{cases}$$



$$\theta_B = \tan^{-1} \sqrt{\frac{1}{80}} \cong 6.38^\circ \quad \text{microwaves}$$

$$\theta_B = \tan^{-1} \sqrt{\frac{1}{1.8}} \cong 36.7^\circ \quad \text{optical}$$

## At the Brewster angle

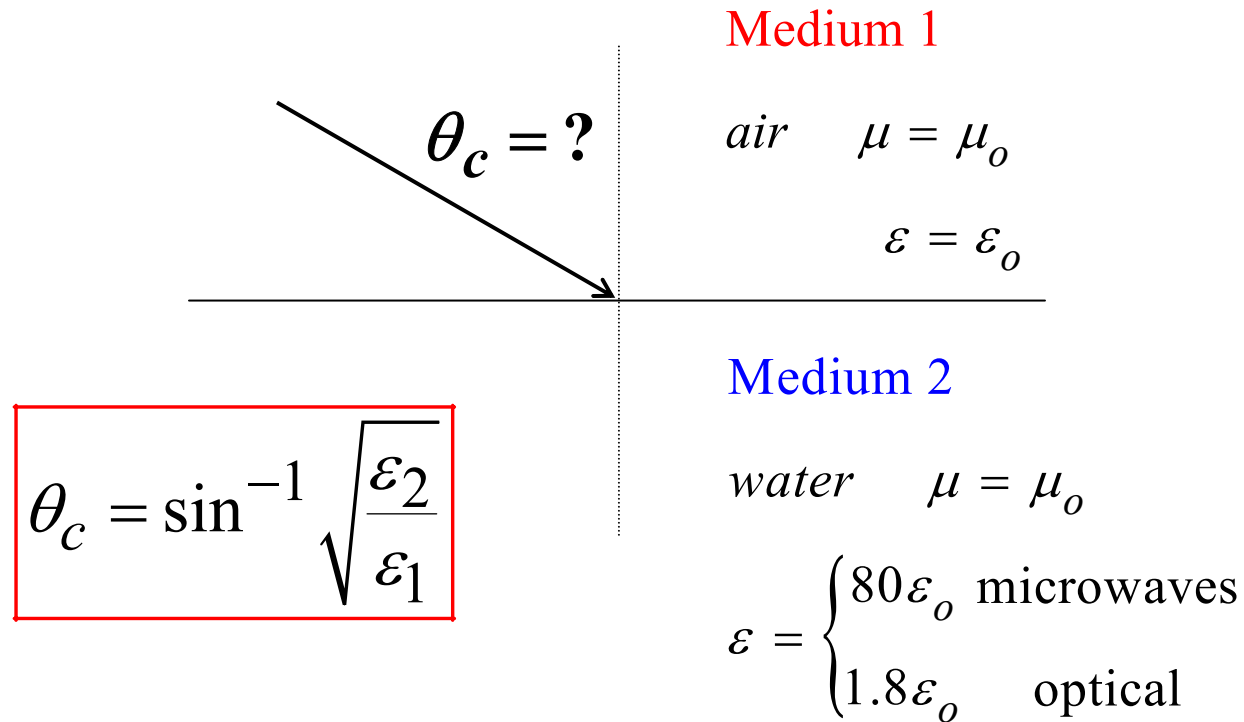
$$\theta_i + \theta_t = \theta_B + \theta_t = 90^\circ$$

$$\Rightarrow \theta_t \cong \begin{cases} 83.6^\circ & \text{microwaves} \\ 53.3^\circ & \text{optical} \end{cases}$$

## Verification with Snell's law

$$\sqrt{\varepsilon_1} \sin \theta_B = \sqrt{\varepsilon_2} \sin \theta_t$$

$$\theta_t = \sin^{-1} \left( \frac{\sin \theta_B \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_0}} \right) \cong \begin{cases} 83.6^\circ & \text{microwaves} \\ 53.3^\circ & \text{optical} \end{cases}$$



**The total reflection angle does not exist since**

$$\epsilon_2 > \epsilon_1$$

## Medium 1

water  $\mu = \mu_o$

$$\varepsilon = \begin{cases} 80\varepsilon_o & \text{microwaves} \\ 1.8\varepsilon_o & \text{optical} \end{cases}$$

$\theta_c = ?$

## Medium 2

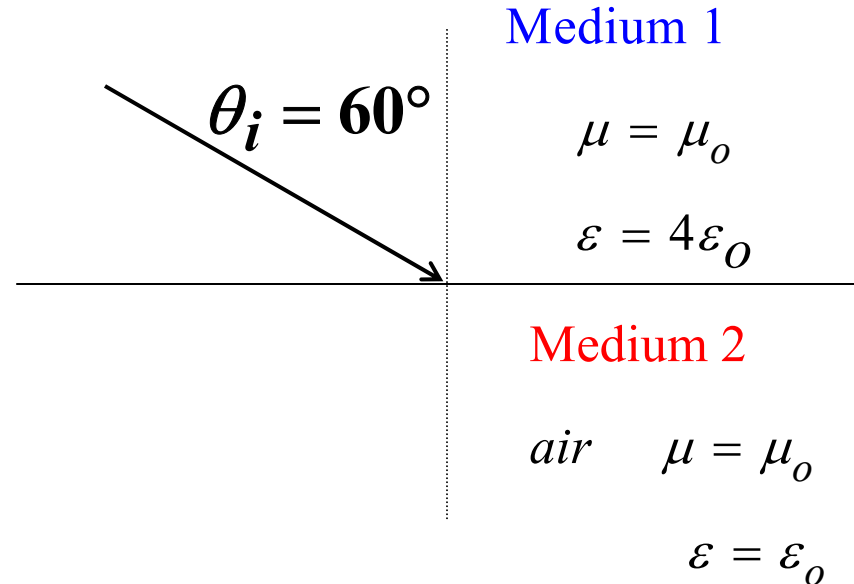
air  $\mu = \mu_o$

$$\varepsilon = \varepsilon_o$$

$$\theta_c = \sin^{-1} \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\theta_c = \sin^{-1} \sqrt{\frac{1}{80}} \cong 6.4193^\circ \quad \text{microwaves}$$

$$\theta_c = \sin^{-1} \sqrt{\frac{1}{1.8}} \cong 48.19^\circ \quad \text{optical}$$



**Consider a perpendicularly polarized wave.**

- **Find the Brewster angle and the critical angle:**

$$\theta_B = \tan^{-1} \sqrt{\frac{1}{4}} = \tan^{-1} \left( \frac{1}{2} \right) \approx 26.565^\circ$$

$$\theta_c = \sin^{-1} \sqrt{\frac{1}{4}} = \sin^{-1} \left( \frac{1}{2} \right) = 30^\circ$$

- Find the components of the incident propagation vector and of the x-component of the transmitted propagation vector in terms of

$$\beta_o = \omega \sqrt{\mu_o \epsilon_o}$$

$$\beta_{ix} = \beta_1 \cos \theta_i = \omega \sqrt{\mu_o 4\epsilon_o} \cos 60^\circ = \frac{2\beta_o}{2} = \beta_o$$

$$\beta_{iz} = \beta_1 \sin \theta_i = \omega \sqrt{\mu_o 4\epsilon_o} \sin 60^\circ = 2\beta_o \frac{\sqrt{3}}{2} = \sqrt{3}\beta_o$$

$$\beta_{tx} = \sqrt{\beta_t^2 - \beta_{tz}^2} \quad \beta_t = \beta_o \quad \beta_{tz} = \beta_{iz} = \sqrt{3}\beta_o$$

$$\beta_{tx} = \underset{\substack{\pm \\ \downarrow \\ \text{choose} \\ \text{"_"}}}{\sqrt{\beta_o^2 - 3\beta_o^2}} = -j\sqrt{2}\beta_o = -j\alpha_t$$

- In the second medium, find the distance at which the field strength is  $1/e$  of that at the interface

$$d = \frac{1}{\alpha_t} = \frac{1}{\sqrt{2}\beta_o}$$

- What is the value of the magnitude of the reflection coefficient at the interface?

The reflection coefficient is a complex quantity when the incident angle exceeds the critical angle. Because of total reflection we know that it must be

$$|\Gamma_{\perp}(E)| = 1$$

since the time-average power reflection coefficient is

$$R = |\Gamma_{\perp}(E)|^2 = 1$$