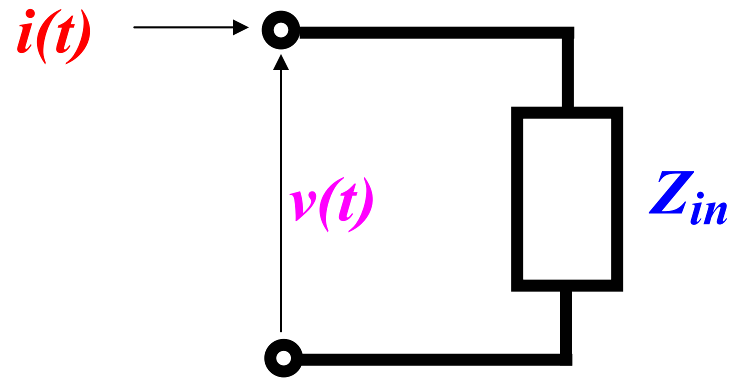
Power in Circuits

Consider the input impedance of a transmission line circuit, with an applied voltage $v(t)$ inducing an input current $i(t)$.



For sinusoidal excitation, we can write

$$v(t) = V_0 \cos(\omega t + \phi) \qquad \phi \in [-\pi/2 \,,\, \pi/2]$$

$$i(t) = I_0 \cos(\omega t)$$

where V_0 and I_0 are peak values and ϕ is the phase difference between voltage and current. Note that $\phi = 0$ only when the input impedance is real (purely resistive).

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The time-dependent input power is given by

$$P(t) = v(t)\, i(t) = V_0\, I_0 \cos(\omega t + \phi) \cos(\omega t)$$

$$= \frac{V_0\, I_0}{2} \left[\cos(\phi) + \cos(2\omega t + \phi) \right]$$

The power has two (Fourier) components:

(A) an average value

$$\frac{V_0\, I_0}{2} \cos(\phi)$$

(B) an oscillatory component with frequency $2f$

$$\frac{V_0\, I_0}{2} \cos(2\omega t + \phi)$$

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The power flow changes periodically in time with an oscillation like (B) about the average value (A). Note that only when $\phi = 0$ we have cos(ϕ) $= 1$, implying that for a resistive impedance the power is always positive (flowing from generator to load).

When voltage and current are out of phase, the average value of the power has lower magnitude than the peak value of the oscillatory component. Therefore, during portions of the period of oscillation the power can be negative (flowing from load to generator). This means that when the power flow is positive, the reactive component of the input impedance stores energy, which is reflected back to the generator side when the power flow becomes negative.

For an oscillatory excitation, we are interested in finding the behavior of the power during one full period, because from this we can easily obtain the average behavior in time. From the point of view of power consumption, we are also interested in knowing the power dissipated by the resistive component of the impedance.

Using $\cos(A+B) = \cos A \cos B - \sin A \sin B$ **one can write**

$$v(t) = V_0 \cos(\omega t + \phi) = \underbrace{V_0 \cos\phi \cos\omega t}_{\text{in phase with current}} - \underbrace{V_0 \sin\phi \sin\omega t}_{\text{in quadrature with current}}$$

This gives an alternative expression for power:

$$P(t) = V_0 \cos(\omega t)\, I_0 \cos(\omega t) \cos(\phi) - V_0 \cos(\omega t)\, I_0 \sin(\omega t) \sin(\phi)$$

$$= \underbrace{V_0 I_0 \cos(\phi) \cos^2(\omega t)}_{\boxed{\text{Real Power}}} - \underbrace{\frac{V_0 I_0}{2} \sin(\phi) \sin(2\omega t)}_{\boxed{\text{Reactive Power}}}$$

$$= \underbrace{\frac{V_0 I_0}{2} \cos(\phi) + \frac{V_0 I_0}{2} \cos(\phi) \cos(2\omega t)}_{\boxed{\text{Real Power}}} - \underbrace{\frac{V_0 I_0}{2} \sin(\phi) \sin(2\omega t)}_{\boxed{\text{Reactive Power}}}$$

The real power corresponds to the power dissipated by the resistive component of the impedance, and it is always positive.

The reactive power corresponds to power stored and then reflected by the reactive component of the impedance. It oscillates from positive to negative during the period.

Until now we have discussed properties of instantaneous power. Since we are considering time-harmonic periodic signals, it is very convenient to consider the time-average power

$$\langle\, P(t)\,\rangle = \frac{1}{T}\int_0^T P(t)\,dt$$

where $T = 1/f$ is the period of the oscillation.

To determine the time-average power, we can use either the Fourier or the real/reactive power formulation.

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Fourier representation

$$\langle\, P(t)\,\rangle = \frac{1}{T}\int_0^T \frac{V_0\, I_0}{2}\cos(\phi)\, dt + \underbrace{\cancel{\frac{1}{T}\int_0^T \frac{V_0\, I_0}{2}\cos(2\omega\, t - \phi)\, dt}}_{=0}$$

$$= \frac{V_0\, I_0}{2}\cos(\phi)$$

As one should expect, the time-average power flow is simply given by the Fourier component corresponding to the average of the original signal.

Real/Reactive power representation

$$\langle P(t) \rangle = \frac{1}{2T} \left(\int_0^T V_0 I_0 \cos(\phi)\, dt + \underbrace{\cancel{\int_0^T V_0 I_0 \cos(\phi) \cos(2\omega t)\, dt}}_{=0} \right)$$

$$- \frac{1}{2T} \underbrace{\cancel{\int_0^T V_0 I_0 \sin(\phi) \sin(2\omega t)\, dt}}_{=0}$$

$$= \frac{V_0 I_0}{2} \cos(\phi)$$

This result tells us that the time-average power flow is the average of the real power. The reactive power has zero time-average, since power is stored and completely reflected by the reactive component of the input impedance during the period of oscillation.

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The maximum of the reactive power is

$$\max\{ P_{reac} \} = \max\{ \frac{V_0 I_0}{2} \sin(\phi) \sin(2\omega t) \} = \frac{V_0 I_0}{2} \sin(\phi)$$

Since the time-average of the reactive power is zero, we often use the maximum value above as an indication of the reactive power.

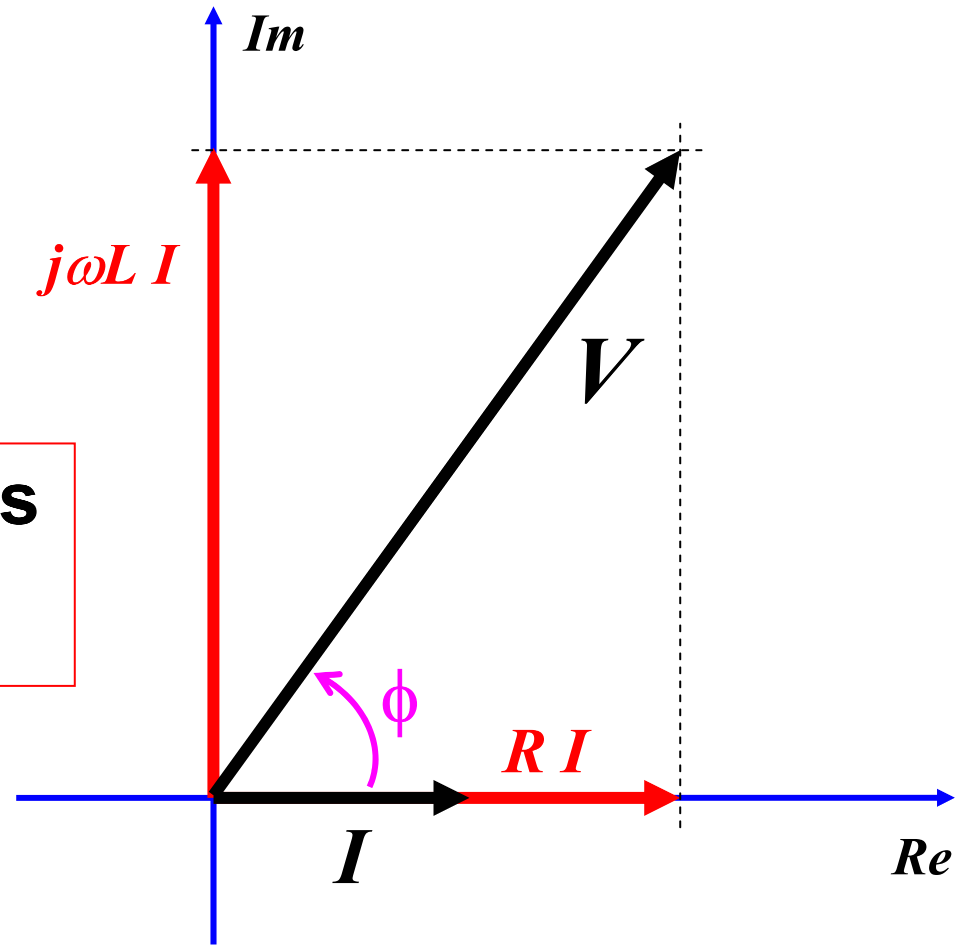
The sign of the phase ϕ tells us about the *imaginary part of the impedance* or *reactance*:

$\phi > 0$ **The reactance is inductive**
Current is lagging with respect to voltage
Voltage is leading with respect to current

$\phi < 0$ **The reactance is capacitive**
Voltage is lagging with respect to current
Current is leading with respect to voltage

If the total reactance is inductive

$$V = Z\, I = R\, I + j\omega L\, I$$

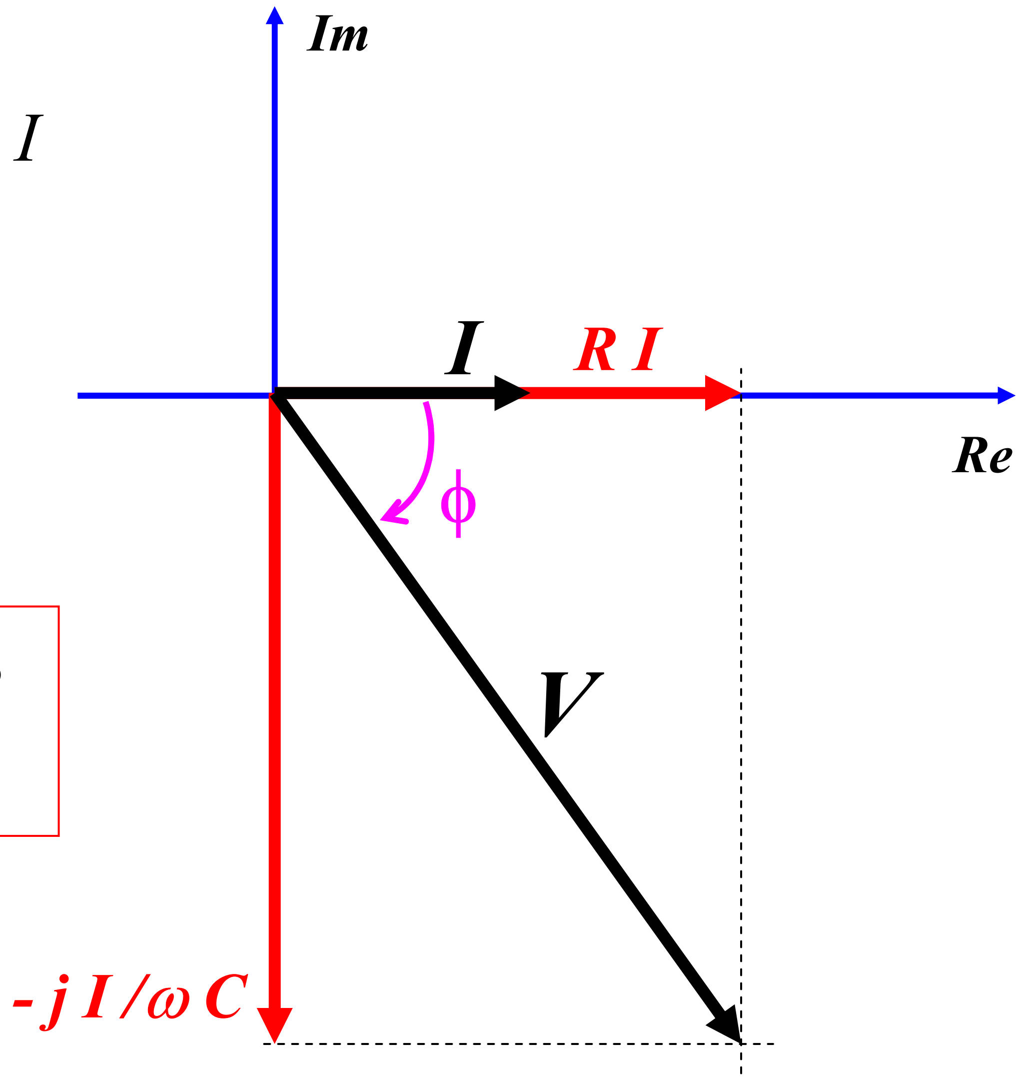


Current lags

$\phi > 0$

If the total reactance is capacitive

$$V = Z\, I = R\, I - j\frac{1}{\omega C}\, I$$



Voltage lags

$\phi < 0$

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In many situations, we may use the root-mean-square (r.m.s.) values of quantities, instead of the peak value. For a given signal

$$v(t) = V_0 \cos(\omega\, t)$$

the r.m.s. value is defined as

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_0^2 \cos^2(\omega t)\, dt} = V_0 \sqrt{\frac{1}{\omega T}\int_0^{2\pi} \cos^2(\omega t)\, d\omega t}$$

$$= V_0 \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \cos^2(\vartheta)\, d\vartheta} = \frac{1}{\sqrt{2}} V_0$$

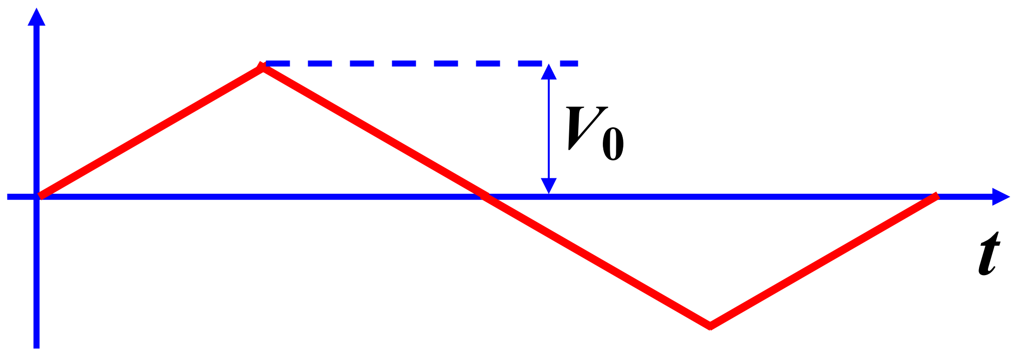
This result is valid for sinusoidal signals. Any given signal shape corresponds to a specific coefficient (peak factor = V_0 / V_{rms}) that allows one to convert directly from peak value to r.m.s. value.

The peak factor for sinusoidal signals is

$$\frac{V_0}{V_{rms}} = \sqrt{2} \approx 1.4142$$

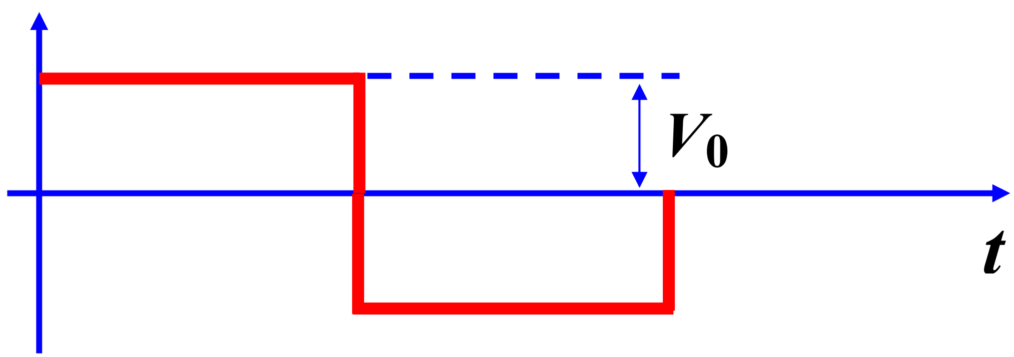
For a symmetric triangular signal the peak factor is

$$\frac{V_0}{V_{rms}} = \sqrt{3} \approx 1.732$$



For a symmetric square signal the peak factor is simply

$$\frac{V_0}{V_{rms}} = 1$$



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For a non-sinusoidal periodic signal, we can use a decomposition into orthogonal Fourier components to obtain the r.m.s. value:

$$\boxed{V(t) = V_{av} + V_1(t) + V_2(t) + V_3(t) \ldots = \sum_k V_k(t)}$$

$$[V_{rms}]^2 = \frac{1}{T}\int_0^T V^2(t)\, dt = \frac{1}{T}\int_0^T [\sum_k V_k(t)]^2\, dt =$$

$$= \frac{1}{T}\int_0^T [\sum_k V_k^2(t)]\, dt + \frac{1}{T}\int_0^T [\sum_{i\neq j} 2\, V_i(t)\, V_j(t)]\, dt =$$

$$= \sum_k \left[\frac{1}{T}\int_0^T V_k^2(t)\, dt\right] + \cancel{\sum_{i\neq j}[\frac{1}{T}\int_0^T 2\, \overbrace{V_i(t)\, V_j(t)}^{\text{orthogonal}}\, dt\,]} = \sum_k (V_{k_{rms}})^2$$

$$\boxed{\Rightarrow V_{rms} = \sqrt{\sum_k (V_{k_{rms}})^2} = \sqrt{V_{av}^2 + (V_{1_{rms}})^2 + (V_{2_{rms}})^2 + \ldots}}$$

The final result holds for any decomposition into orthogonal functions and it is known in mathematics as Parseval's identity.

In terms of r.m.s. values, the time-average power for a sinusoidal signal is then

$$\langle P(t) \rangle = \frac{V_0}{\sqrt{2}} \frac{I_0}{\sqrt{2}} \cos(\phi) = V_{rms}\, I_{rms} \cos(\phi)$$

Finally, we can relate the time-average power to the phasors of voltage and current. Since

$$v(t) = V_0 \cos(\omega t) = \mathrm{Re}\{\, V_0 \exp(j\omega t)\,\}$$

$$i(t) = I_0 \cos(\omega t - \phi) = \mathrm{Re}\{\, I_0 \exp(-j\phi) \exp(j\omega t)\,\}$$

we have phasors

$$V = V_0$$

$$I = I_0 \exp(-j\phi)$$

The time-average power in terms of phasors is given by

$$\langle P(t) \rangle = \frac{1}{2}\mathrm{Re}\{ V\, I^{*} \} = \frac{1}{2}\mathrm{Re}\{ V_0\, I_0 \exp(j\phi) \}$$

$$= \frac{V_0\, I_0}{2}\cos(\phi)$$

Note that one must always use the complex conjugate of the phasor current to obtain the time-average power. It is important to remember this when voltage and current are expressed as functions of each other. Only when the impedance is purely resistive, $I = I^* = I_0$ since ϕ = 0.

Also, note that the time-average power is always a real positive quantity and that it is not the phasor of the time-dependent power. It is a common mistake to think so.

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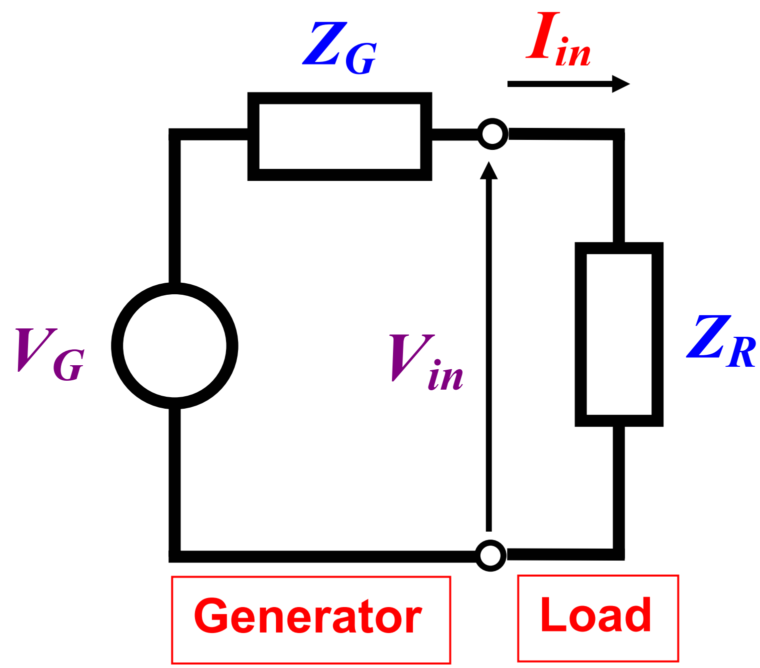
Now we consider power flow including explicitly the generator, to understand in which conditions maximum power transfer to a load can take place.

$$V_{in} = V_G \frac{Z_R}{Z_G + Z_R}$$

$$I_{in} = V_G \frac{1}{Z_G + Z_R}$$

$$\langle P_{in} \rangle = \frac{1}{2} \mathrm{Re}\{ V_{in}\, I_{in}^{*} \}$$



As a first case, we examine resistive impedances

$$Z_G = R_G \qquad\qquad Z_R = R_R$$

Voltage and current are in phase at the input. The time-average power dissipated by the load is

$$\langle\, P(t)\,\rangle = \frac{1}{2} V_G \frac{R_R}{R_G + R_R} V_G^* \frac{1}{R_G + R_R}$$

$$= \frac{1}{2} \left| V_G \right|^2 \frac{R_R}{(R_G + R_R)^2}$$

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To find the load resistance that maximizes power transfer to the load for a given generator we impose

$$\frac{d\langle P(t)\rangle}{d\,R_R} = 0$$

from which we obtain

$$\frac{d}{dR_R}\left[\frac{R_R}{(R_G + R_R)^2}\right] = 0$$

$$\frac{(R_G + R_R)^2 - 2R_R(R_G + R_R)}{(R_G + R_R)^4} = 0$$

$$(R_G + R_R) - 2R_R = 0 \qquad \Rightarrow \qquad R_R = R_G$$

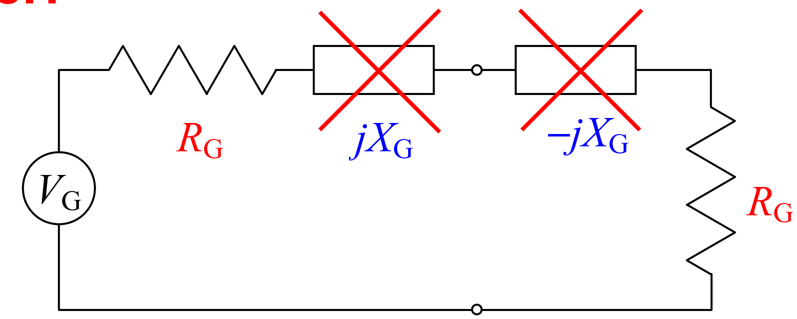
We conclude that for maximum power transfer the load resistance must be identical to the generator resistance.

Let's consider now complex impedances

$$Z_R = R_R + jX_R \qquad\qquad Z_G = R_G + jX_G$$

For maximum power transfer, generator and load impedances must be complex conjugate of each other:

$$Z_R = Z_G^* \quad \Rightarrow \quad R_R = R_G$$
$$X_R = -X_G$$



This can be easily understood by considering that, to maximize the active power supplied to the load, voltage and current of the generator should remain in phase. If the reactances of generator and load are opposite and cancel each other along the path of the current, the generator will only see a resistance. Voltage and current will be in phase with maximum power delivered to the load.

The total time-average power supplied by the generator in conditions of maximum power transfer is

$$\langle P_{tot} \rangle = \frac{1}{2} \mathrm{Re}\{ V_G \; I_{in}^{*} \} = \frac{1}{2} |V_G|^2 \frac{1}{2R_R} = \frac{1}{4} |V_G|^2 \frac{1}{R_R}$$

The time-average power supplied to the load is

$$\langle P_{in} \rangle = \frac{1}{2} \mathrm{Re}\{ V_{in} \; I_{in}^{*} \} = \frac{1}{2} \mathrm{Re}\left\{ V_G \frac{Z_R}{Z_G + Z_R} V_G^{*} \left(\frac{1}{Z_G + Z_R} \right)^{*} \right\}$$

$$= \frac{1}{2} |V_G|^2 \; \mathrm{Re} \left\{ \frac{R_R + jX_R}{4R_R^2} \right\} = \frac{1}{8} |V_G|^2 \frac{1}{R_R}$$

The power dissipated by the internal generator impedance is

$$\langle P_G \rangle = \frac{1}{2} \operatorname{Re}\{ (V_G - V_{in})\, I_{in}^{*} \}$$

$$= \frac{1}{4} |V_G|^2 \frac{1}{R_R} - \frac{1}{8} |V_G|^2 \frac{1}{R_R} = \frac{1}{8} |V_G|^2 \frac{1}{R_R}$$

We conclude that, in conditions of maximum power transfer, only half of the total active power supplied by the generator is actually used by the load. The generator impedance dissipates the remaining half of the available active power.

This may seem a disappointing result, but it is the best one can do for a real generator with a given internal impedance!